THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE seventh regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 25, 1905. The following members were present:

Professor H. F. Blichfeldt, Professor G. C. Edwards, Mr. G. I. Gavett, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor A. O. Leuschner, Professor G. A. Miller, Professor H. C. Moreno, Dr. B. L. Newkirk, Professor C. A. Noble, Professor T. J. J. See, Professor Irving Stringham, Professor A. W. Whitney.

The attendance also included a number of teachers of mathematics and physics who are not members of the Society. A morning and an afternoon session were held, Professor Haskell acting as chairman at both sessions. It was agreed to hold the next meeting at the University of California on September 30, 1905.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "On a theorem due to C. Jordan."

(2) Professor H. F. BLICHFELDT: "On the order of the collineation groups in five variables."

(3) Professor A. W. WHITNEY: "A theorem in the theory of probabilities, and its application to insurance."

(4) Professor R. E. MORITZ: "A general theorem on local probability."

(5) Professor E. J. WILCZYNSKI: "Projective differential geometry of plane curves."

(6) Dr. W. A. MANNING: "On the primitive groups of class ten."

(7) Professor G. A. MILLER: "Invariant subgroups of prime index."

(8) Professor IRVING STRINGHAM: "A geometric construction for quaternion products."

(9) Professor A. O. LEUSCHNER: "On the general applicability of the short method of determining orbits from three observations."

(10) Professor T. J. J. SEE: "On the physical state of the matter of the earth's interior, with considerations on terrestrial geology, and on the comparative geology of the other planets."

In the absence of the authors, the papers of Professors Moritz and Wilczynski were read by Professors Leuschner and Haskell respectively; Dr. Manning's paper was presented by Professor H. C. Moreno. Professor Stringham's paper will appear in a later number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. The theorem due to C. Jordan, that the order of a linear homogeneous group G in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup of G, and where γ is inferior to a fixed number depending only upon n, has been proved for a class of linear homogeneous groups called primitive by Professor Blichfeldt (*Transactions*, volume 4, page 387; volume 5, page 310), a number having been found that λ must divide. In the present paper, Professor Blichfeldt proved the corresponding theorem for imprimitive groups of the kind considered, thus completing his proof of Jordan's theorem for all transitive ("irreducible") groups, from which the theorem is readily demonstrated for groups in general.

2. Professor Blichfeldt's second paper is devoted to a proof of the following theorem: In a primitive group G in 5 variables, containing no invariant subgroup and not being identical with the group H (cf. *Transactions*, volume 5, page 315), the number λ must be a factor of one of the following numbers: $2^7 \cdot 3^5 \cdot 5^5$, $2^7 \cdot 3^3 \cdot 5^3 \cdot k$, where k is one of the numbers 7^2, $7 \cdot 11$, 11^2, $11 \cdot 13$, 13^2, 17 or 19.

3. Professor Whitney proposed the ratio of the expected loss to the probable fluctuation as an index of the security of the profits from being used in meeting exceptional losses. From this index he calculated the actual probability that they will be thus exhausted in any year. The condition for this index to be a maximum yields a criterion for the best distribution of lines. The method readily lends itself to the consideration of partial loss. It is furthermore entirely susceptible of purely statistical treatment.

4. Czuber, in his Geometrische Wahrscheinlichkeiten und Mittelwerte, shows that if on a straight line of length a two points are assumed at random, the probability that their dis-

tance apart exceeds a given length b is $P = [(a-b)/a]^2$. Professor Moritz extends this theorem to the case of the probability that of n points distributed at random along a line of length a, no two shall fall within a distance b of each other. If $x_1, x_2, \cdots, x_n$ represent the respective distances of n points from one extremity of the line a, and if Q represents the probability that no two points fall within a distance b of each other when the points are subject to the conditions $x_1 < x_2 < \cdots < x_n$, then $P = n!\, Q$, where

$$Q = \frac{1}{n!}\left(\frac{a-\overline{n-1}b}{a^2}\right)^n, \quad P = \left(\frac{a-\overline{n-1}b}{a}\right)^n.$$

For the case of three points the theorem can be proven geometrically. One set of concurrent axes of a cube with edge a are taken as axes of coördinates. A one to one correspondence is seen to exist between the coördinates x, y, z of any point within the cube and the respective distances x_1, x_2, x_3 of three points on a line of length a. The problem now consists in comparing the volume of the entire cube with the volume of the regions for which simultaneously

$$x \sim y > b, \quad y \sim z > b, \quad z \sim x > b.$$

This gives for the required probability $P = [(a-2b)/a]^3$.

From the general proportion follows the important corollary: The probability that of n points, distributed at random on a closed curve of length a, no two shall fall within an arc b of each other is $[(a-nb)/a]^{n-1}$.

5. Professor Wilczynski presents in a simpler and clearer form the results of Halphen's thesis "Sur les invariants différentiels." The paper forms at the same time the conclusion of the author's investigations on differential projective geometry, so far as it concerns curves and ruled surfaces. All of the papers which touch upon this subject will be shortly combined into a systematic treatise on the projective differential geometry of curves and ruled surfaces. It is the author's intention after the completion of this treatise to take up the general theory of surfaces from the same point of view, thus creating for projective geometry a general theory which shall correspond to the place filled by the Gauss-Monge theory in metrical geometry.

Although primarily a new rendering of Halphen's results, this paper is in form entirely different from Halphen's thesis,

and contains many results and points of view not to be found in Halphen's memoir. Still, so far as it has been possible to do so with the different treatment adopted, the author has confined himself to the scope of Halphen's thesis, so as to make more convenient the comparison of the two methods.

6. Four of the primitive groups of class 10 contain substitutions of order 5 on 10 letters. These are the metacyclic group of degree 11, and the Mathieu group of degree 12, and their positive subgroups. This result was given by Dr. Manning in volume 4 of the *Transactions*. In the present paper he completes the list of the primitive groups of class 10 by adding three groups which do not contain this substitution of order 5. One is the diedral rotation group of order 22, one the G_{21}^{21}, simply isomorphic to the symmetric group of degree 7, and one is a new group of degree 25 and order $2(5!)^2$. This last forms part of a system of primitive groups of degree k^2, order $2(k!)^2$, and class $2k$ $(k > 2)$.

7. The number of invariant subgroups of index p in any group G is of the form $(p^\lambda - 1)/(p-1)$. When G is abelian λ is equal to the number of invariants in the Sylow subgroup of order p^m contained in G. The determination of all the possible groups of order p^m in which λ has a given value is quite difficult except in a few cases. When $\lambda = 1$ such a group is cyclic and when $\lambda = m$, G is the abelian group of type $(1, 1, 1, \cdots)$. All the possible groups in which $\lambda = m - 1$ are also known. Professor Miller's paper is devoted to the case where $\lambda = m-2$. The investigations are simplified by the use of the theorem that only invariant operators can be common to every subgroup of index p of a group of order p^m whose commutator subgroup is of order p. The paper will be offered to the *Transactions* for publication.

9. After a brief presentation of the fundamental principles of his "Short method of determining orbits from three observations" without hypothesis regarding the eccentricity, Professor Leuschner developed the formulæ necessary to determine whether, in a given case, the solution is determinate, indeterminate, or partially determinate. He discussed the special cases to which one or the other of the older methods are inapplicable. In particular, it was pointed out that with the "short method" the orbit does not necessarily become indeter-

minate when the three geocentric places and the second solar place lie in a great circle, the following three additional conditions being necessary to make the solution indeterminate in this case: (1) The intervals must be equal; (2) the third powers of the intervals multiplied by the solar constant k must be numerically ineffective; and (3) the heliocentric distance must be comparatively large.

The "short method" is particularly suitable for correcting preliminary elements by means of a longer arc. For this purpose, the geocentric distance and the velocities in the heliocentric rectangular coördinates are first derived for the middle date from the ephemeris or from the constants of the preliminary orbit and then corrected on the basis of the residuals of the first and third observations. Modifications of the "short method" were given to enable a computer to derive (1) the range within which the period of the orbit may lie, and (2) a parabola directly in case the parabola should be included within this range.

10. The purport of this paper was to show that the matter within the earth and other large planets is genuine gas above the critical temperature of every known substance, yet so compressed as to behave with more perfect elasticity than any solid. This view had been reached by the study of the pressures within the other planets, which Professor See has treated in *Astronomische Nachrichten* no. 3992. He submitted calculations to prove that under the immense pressures existing in the heavenly bodies porosity and interpenetrability are general properties of matter, and that all matter thus becomes fluid, the molecular forces entirely disappearing in comparison with the strains to which the matter is subjected. He also showed that if the planets were made of the stiffest granite or steel they would assume the globular form from pressure alone. The spheroidal figures of the planets not only prove their original fluidity, but the fluidity of all matter under planetary pressure, which is reckoned in millions of atmospheres.

As regards the earth he showed that at a depth of 63.7 kms. matter released from the imposed pressure would be crushed, fused and vaporized in the process, and in this way explained volcanic phenomena. Though the earth's matter behaves as a solid, on account of its imprisonment, it is really elastic vapor reduced to great hardness by pressure, and if the pressure were

relieved, as by sinking a vacant shaft down to it, a violent explosion would follow.

The author concluded from Laplace's law of density that the planet Venus has a rigidity equal to that of a corresponding globe of glass, and that the nuclei of all large bodies are effectively of the highest rigidity. The development of pressure as we decend in the globe was shown to be such as to invalidate the conclusions of Lord Kelvin and Professor G. H. Darwin respecting the consolidation of earth by the building up of a solid nucleus from the sinking of the solidified crust.

G. A. MILLER,
Secretary of the Section.

ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATION TO CERTAIN OTHER SCIENCES.*

ADDRESS DELIVERED BEFORE THE SECTION OF ALGEBRA AND ANALYSIS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCES, ST. LOUIS, SEPTEMBER 22, 1904.

BY PROFESSOR EMILE PICARD.

ONE of the objects of such a congress as that in which we are now assembled is to show the connection between the different parts of science taken in its widest sense. Moreover, the promoters of this meeting have insisted that the relations between different branches should be put in evidence. To undertake a study of this kind, the character of which is somewhat indefinite, one must forget that all is in all; as for algebra and analysis alone, a Pythagorean would be dismayed at the extent of his task, remembering the celebrated formula of the school: "Things are numbers." From this point of view, my subject would be inexhaustible. But for excellent reasons I should not attempt so much. Merely glancing at the development of our science through the ages, and particularly in the last century, I hope to be able to characterize sufficiently the

* Translated, with the author's permission, by Professor M. W. HASKELL.

rôle of mathematical analysis in its relation to certain other sciences.

I.

It would appear natural to begin by speaking of the very idea of whole number; but this subject belongs not only to logic, but to history and psychology, and would involve us in too many discussions. In the investigation of the number concept, unfathomable depths have revealed themselves. Thus, it is a problem still unsolved as between the two forms, cardinal and ordinal, under which the idea of number presents itself, to know which of the two is anterior to the other; that is to say, whether the idea of number properly so called is anterior to that of order, or inversely. In these questions, the geometric logicians appear to pay too little attention to psychology, and to the information furnished us by uncivilized peoples. Perhaps, too, there is no general answer to the question proposed, this answer varying according to race and mental endowment. I have sometimes thought, apropos of this subject, of the distinction between those in whom the auditory or the visual sense predominates, the former adhering to the ordinal theory and the latter to the cardinal theory. But I shall not waste time on this ground so full of snares; I fear that our modern school of logicians has difficulty in coming to an understanding with the ethnographers and biologists; the latter are always dominated by the doctrine of evolution in questions of origin, and more than one of them regards logic as only the résumé of ancestral experience. Mathematicians have even been reproached with laying down as a principle that there is a human intellect somehow exterior to things, and that it has its own logic. Whatever may be thought of this idea, it has proven useful, nay almost indispensable, in the progress of mathematics; and certainly, supposing that it evolved in the course of prehistoric times, this logic of the human intellect was well established at the time of the early geometric schools among the Greeks; and their works would appear to have been its first code, as is expressed by the story of Plato's writing over the entrance to his school: "Let no man enter here who is not a geometer."

Long before the acquisition from the Arabs of the extraordinary word, algebra, which, it appears, expresses the operation by which equalities are reduced to a certain canonical form, the Greeks had employed algebra without knowing it; one

cannot even imagine closer relations than those connecting their algebra and their geometry, or rather, one would be at a loss to classify, if there were occasion, their geometric algebra, in which they reasoned not only about numbers, but also about quantities. Among the Greeks we find also a geometric arithmetic, and one of the most interesting phases of its development is the conflict which arose among the Pythagoreans on this subject between number and quantity apropos of irrationals. While the Greeks cultivated the abstract study of numbers, under the name of arithmetic, their speculative spirit showed less taste for practical calculation, which they called logistic. In the earliest times, the Egyptians and Chaldees, and later the Hindoos and Arabs developed the science of computation. They had been led to it by practical necessities; logistic preceded arithmetic as surveying and geodesy paved the way for geometry; in the same way, moreover, trigonometry was developed by the increasing needs of astronomy. The history of science in its beginnings shows a close relation between pure mathematics and applied mathematics; we shall find this relation again and again as we proceed.

We have remained so far in the domain called in ordinary language elementary algebra and arithmetic; but as soon as the incommensurability of certain quantities was recognized, the notion of infinity made its appearance, and after the time of the paradoxes of Zeno on the impossibility of movement, the summation of geometric progressions could not be avoided. The methods of exhaustion found in Eudoxus and in Euclid properly belong to the integral calculus, and Archimedes calculates definite integrals. Mechanics appears also in his treatise on the quadrature of the parabola, for he finds the surface of the segment bounded by a parabolic arc and its chord at first by depending on the theorem of moments; it is the first example of the relations between mechanics and analysis whose development has been continuous since that time. The infinitesimal method of the Greek geometricians on the measure of volumes has raised questions whose interest is not exhausted even to-day. In plane geometry, two equivalent polygons are equal by addition or subtraction, that is to say, can be decomposed into equal triangles, or be regarded as differences of polygons which can be so decomposed. It is not the same for the geometry of space, and we have recently learned that the equivalence of solid polyhedra, unlike that of plane polygons, can

only be established by recourse to the method of exhaustions or to that of limits, which require the axiom of continuity, or the axiom of Archimedes. Without laying any more stress on this point, this glance cast at antiquity shows how closely algebra, arithmetic, geometry, and the first attempts in integral calculus and in mechanics were intermingled at that time, so closely indeed that it is impossible to relate their history separately.

During the Middle Ages and the Renaissance the geometric algebra of the ancients is separated from geometry. Little by little algebra properly so called arrives at autonomy, with its symbolism and its notation more and more perfected; thus is created this language of a wonderful clearness, which provides a true economy of thought and renders further progress possible. This is the moment when distinct divisions are organized. Trigonometry, which in antiquity had been only an auxiliary of astronomy, is developed into an independent subject; at about the same time logarithms made their appearance, and essential elements were thus put in evidence.

II.

In the seventeenth century the analytic geometry of Descartes, different from what I have just called the geometric algebra of the Greeks in the general and systematic ideas upon which it is founded, and the newly-born science of dynamics, were the origin of very great advances in analysis. When Galileo, starting from the hypothesis that the velocity of heavy bodies in their fall is proportional to the time, deduced from this the law of the distance traversed, and verified it afterwards by experiment, he took up again the path where Archimedes had traveled long before, and upon which he was to be followed by Cavalieri, Fermat, and others still down to the time of Newton and Leibniz. The integral calculus of the Greek geometricians was born again in the kinematics of the great Florentine physicist. As to the calculus of derivatives or of differentials, it was stated with precision apropos of the direction of tangents. In reality the origin of the notion of derivatives is in the vague feeling of the mobility of things, and of the greater or less speed with which phenomena take place; this is well expressed by the terms fluent and fluxion, which were used by Newton and which we may believe were borrowed from the ancient mathematician Heraclitus.

The points of view taken by the founders of the science of motion, Galileo, Huyghens, and Newton, had an enormous influence on the orientation of mathematical analysis. In Galileo it was a stroke of genius to perceive that in natural phenomena the determining circumstances of the movement produce accelerations; this led necessarily to the formulation of the principle that the speed with which the dynamic state of a system changes depends in a definite way upon its static condition alone. In a more general way, it came to be postulated that infinitesimal changes, no matter what their nature, which occur in a system of bodies depend solely on the present state of the system. In how far are the exceptions apparent or real? This is a question which was only raised later and which I waive for the moment. From among the principles enunciated an important point for the analyst presents itself; phenomena are governed by differential equations, equations which can be formed when observation and experiment have made known certain physical laws for each category of phenomena. We understand the unbounded hopes to which these results must have given rise. As Bertrand says in the preface to his Traité, "The first successes were such that one might suppose all the difficulties of science overcome in advance, and believe that the mathematician, without being longer occupied in the elaboration of pure mathematics, could turn his thoughts exclusively to the study of natural laws." This was taking for granted that the problems of analysis presenting themselves would offer no very serious difficulties. In spite of the disillusions which the future had in store, there remained this important point, that the problems had taken a precise form, and that the difficulties yet to be overcome could be classified. This constituted an immense advance, one of the greatest which the human mind has ever made. We observe also the reason why the theory of differential equations was destined to acquire a great importance.

I have anticipated somewhat in presenting things under so analytic a form. Geometry was involved in all of these advances. Huyghens, for instance, preferred to follow the ancients, and his Horologium oscillatorium depends both on infinitesimal geometry and on mechanics; likewise, the methods followed in Newton's Principia are synthetic. It is primarily with Leibniz that science begins to follow the paths which were to lead to what we call mathematical analysis; he it is who for the first time, in the later years of the seventeenth century, pronounces

the word function. Through his systematic mind, through the numerous problems treated by him, as well as by his pupils, James and John Bernoulli, he established in a definite way the power of methods, to the building up of which a long series of thinkers had contributed, one after another, ever since the remote times of Eudoxus and Archimedes.

The eighteenth century shows the extreme fruitfulness of this new method. It was a curious time, that, of those mathematical duels, when geometers hurled defiances at each other, struggles which were not always without bitterness, when Leibnizians and Newtonians met in the arena. From the purely analytic point of view, the classification and study of simple functions is particularly interesting; the idea of function upon which analysis rests is thus developed little by little. The celebrated works of Euler occupied an important place at that time. However, the numerous problems which were presented to mathematicians hardly left them time to scrutinize the principles; the theoretical foundations themselves are slowly cleared up, and the saying attributed to d'Alembert: "Go forward and faith will come to you," is thoroughly characteristic of this epoch. Of all the problems raised at the end of the seventeenth century or during the first half of the eighteenth century, it is sufficient for me to recall those isoperimetric problems which were to give birth to the calculus of variations. I prefer to emphasize the more intimate relations still between analysis and mechanics, when after the inductive period of the early years of dynamics the deductive period arrived, in which the effort was made to give a definitive form to the principles. The mathematical and formal development then played the essential rôle, and the analytic language was indispensable to the greatest extension of these principles. There are moments in the history of sciences, and perhaps of societies, in which the mind is sustained and carried forward by the words and symbols which it has created, and in which generalizations present themselves with the least effort. Such was particularly the rôle of analysis in the formal development of mechanics. Let me here be permitted one remark. It is often said that an equation contains only what has been put into it. It is easy to reply in the first place that the new form under which things are found often constitutes by itself alone an important discovery. But there is something more: analysis, by the simple play of its symbols, may suggest generalizations far beyond the original

limits. Is it not so with the principle of virtual velocities, of which the first idea came from the simplest mechanisms? The analytic form into which it is translated suggested extensions which led far from the point of departure. Similarly, it is not right to say that analysis has created nothing, since these more general conceptions are its work. Another example is furnished us by the system of Lagrange's equations; here calculus transformations have given the type of differential equations to which one tends to refer to-day the notion of mechanical explanation. There are in science few examples comparable with this of the importance of the form of an analytic relation and of the power of generalization of which it can be capable. It is very clear that in every case the generalizations suggested must be stated precisely by an appeal to observation and experiment; finally, it is still the calculus which will deduce the remote consequences to be submitted to the same tests, but this is an order of ideas which I am not to consider here.

Under the impulse of the problems suggested by geometry, mechanics and physics, we see almost all of the great divisions of analysis develop or originate. In the first place we met equations with a single independent variable. Soon equations with partial derivatives were to appear in connection with vibrating strings, the mechanics of fluids and the infinitesimal geometry of surfaces. This was all a new analytic world; the very origin of the problems treated was a help to prevent our going astray in the first steps, and in the hands of Monge geometry rendered useful service to the newly born theories. But of all the applications of analysis, no one was more brilliant than the problems of celestial mechanics suggested by the knowledge of the laws of gravitation, with which are associated the names of the greatest mathematicians. Theory never had a finer triumph; perhaps one might add that it was too complete, for this is the very moment above all in which were conceived for natural philosophy those hopes, premature to say the least, of which I spoke above. In all this period, especially in the second half of the eighteenth century, what strikes us with admiration, and at the same time introduces some confusion, is the extreme importance of the applications realized, while the pure theory appeared still so insecure. We see this when certain questions are raised like that of the degree of arbitrariness in the integral of vibrating strings, which gives rise to a protracted discussion still far from being concluded. Lagrange

felt these insufficiencies when he published his theory of analytic functions, in which he endeavors to give a precise foundation to analysis. We cannot too much admire the marvellous presentiments which he had of the part to be played by the functions which we call, as he did, analytic; but we may admit our amazement at the demonstration which he thought he had given of the possibility of the development of a function in Taylor's series. The requirements in the questions of pure analysis were less at this epoch. Trusting to intuition, mathematicians took up with certain plausibilities and depended implicitly upon certain hypotheses which it appeared to them useless to formulate explicitly; at bottom, they trusted in the solidity of ideas which had been so many times proved fruitful; this is pretty nearly the well known saying of d'Alembert. The need of rigor in mathematics has had its successive approximations, and in this respect our science has not the absolute character so many persons attribute to it.

III.

We have now arrived at the first years of the nineteenth century. We have explained how the great majority of the analytic investigations in the eighteenth century were occasioned by geometric, and particularly by mechanical and physical problems; and we have found scarcely a trace of the logical and æsthetic predispositions which have given such a different physiognomy to so many mathematical works, especially of the last two-thirds of the nineteenth century. Let us not anticipate, however. After so numerous examples of the influence of physics on the development of analysis, we shall meet still others, and those of the utmost importance, in Fourier's theory of heat.

Fourier begins by forming the partial differential equations which govern the temperature. What are the boundary conditions which allow the determination of a solution of a partial differential equation? With Fourier, the conditions are suggested by the physical problem, and the methods which he followed served as models for the geometric physicists of the first half of the last century. One of them consists in forming series of certain simple solutions. Fourier obtained in this way the first types of more general developments than the trigonometric; as in the problem of the cooling of a sphere, where he applies his theory to the terrestial globe, and investigates

the law governing the variations of temperature in the sun, endeavoring to obtain numerical applications. In the face of so many beautiful results, we understand the enthusiasm which glows in every line of Fourier's introductory chapter. Speaking of mathematical analysis, "There cannot be," says he, "a language more universal and more simple, more devoid of errors and of obscurities, that is to say more suitable for the expression of the invariable relations of natural objects. Considered from this point of view, it is as wide as nature itself; it defines all perceptible relations, measures time and space, forces and temperatures; this difficult science is formed slowly, but it maintains all the principles that it has once acquired. It increases and strengthens without ceasing in the midst of so many errors of the human intellect." The eulogy is magnificent; but between the lines we read the tendency to regard analysis solely as the helpmeet, however incomparable, of natural science, a tendency in conformity, as we have seen, with the development of science during the two centuries preceding; but we are just coming to the period in which new tendencies were to make their appearance. Poisson having recalled, in a report on the *Fundamenta*, Fourier's complaint about Abel and Jacobi that they were not primarily interested in the movement of heat, Jacobi writes to Legendre: "It is true that M. Fourier was of the opinion that the chief end of mathematics was public utility and the explanation of natural phenomena; but a philosopher like him ought to have known that the sole end of science is the honor of the human intellect, and that under this head a problem of number is as important as a problem of the system of the world." This was also without doubt the opinion of the great mathematician of Göttingen, who called mathematics the queen of sciences and arithmetic the queen of mathematics. It would be absurd to put these two tendencies in opposition to each other; the harmony of our science lies in their synthesis. The moment had to come when the necessity would be felt of inspecting the foundations of the structure and of making an inventory of the accumulated stores of wealth by the introduction of more of the critical spirit. Mathematical thought was to acquire more strength by doubling on its track; problems become exhausted for a time, and it is not good when all the investigators remain in the same road. Besides, unexplained difficulties and paradoxes made advances in the pure theory necessary. The road along which it must move was indicated

in a general way, and therein it could proceed independently without necessarily losing contact with the problems proposed by geometry, mechanics, and physics. At the same time, more interest was going to be attached to the philosophic and artistic side of mathematics, through a trust in a sort of preëstablished harmony between our logical and æsthetic satisfactions and the needs of future applications.

Let us briefly call to mind some points in the history of the revision of the principles in which Gauss, Cauchy, and also Abel were the pioneers.

Precise definitions for continuous functions and their most immediate properties and simple rules for the convergency of series were formulated, and soon the possibility of trigonometric developments was established under very general conditions, thus justifying Fourier's boldness. Certain geometric intuitions respecting areas and arcs gave place to rigorous demonstrations. The geometricians of the eighteenth century necessarily tried to take account of the degree of generality in the solution of ordinary differential equations. The comparison with difference-equations easily led to the result, but this kind of proof will not stand a close inspection. Lagrange, in his lessons on the calculus of functions, introduced more precision, and starting from Taylor's series, he saw that an equation of order m leaves undetermined the function and its first $m - 1$ derivatives for the initial value of the variable; we are not surprised that Lagrange had not considered the question of convergency. In twenty or thirty years the requirements in the rigor of proofs had increased. We know that the two preceding modes of proof are susceptible of all the precision necessary. With the first, no new principle was needed; with the second, the theory had to be developed in a new way.

Up to this point, functions and variables had remained real. By the consideration of complex variables the field of analysis was extended. The functions of a complex variable with a unique derivative are necessarily developable in a Taylor's series; we thus fall back on the method of development whose interest was understood by the author of the theory of analytic functions, but whose importance could not be fully exhibited so long as the investigation was limited to real variables. Moreover, they owe the great part which they still continue to play to the facility with which they can be handled, and to their convenience in calculation. The general theorems in the

theory of analytic functions have permitted the giving of an exact reply to questions previously undecided, like the degree of generality of the integrals of differential equations. It became possible to complete the demonstration sketched by Lagrange for an ordinary differential equation; and for the case of a partial differential equation, or of a system of such equations, precise theorems have been established. Not that the results obtained on this last point, important as they are, solve completely the different questions which can be brought forward; for in mathematical physics, and even in geometry, the boundary conditions admit such a variety of forms that the problem called by Cauchy's name often appears in quite a narrow form. I shall return to this important point presently.

IV.

Without adhering to the historical order, let us return to the development of mathematical physics in the last century, in so far as it concerns analysis. The problems of thermal equilibrium led to the equation already encountered by Laplace in the study of attraction. Few questions have been the subject of so much labor as this famous equation. The boundary conditions may be of various forms. The simplest case is that of the thermal equilibrium of a body whose surface elements are constant — the temperatures given; from the physical point of view, it may be regarded as evident that the temperature, being continuous in the interior, since there is no source of heat, is determined when it is given at the surface. A more general case is that in which, the condition remaining permanent, there would be radiation outward with an intensity varying at the surface according to a given law; in particular, the temperature may be given over one portion of the surface while there is radiation over the remainder. These questions, which have not yet been solved in the most general case, have contributed enormously to the orientation of the theory of partial differential equations. They have called attention to types of determination of the integrals which would not have presented themselves if the point of view had remained purely abstract. Laplace's equation had already been met in hydrodynamics, and in the study of attraction in the inverse ratio of the square of the distance. This last theory has led to the putting in evidence of the most essential elements, like the potential of simple layers and of double layers. There have appeared in

this theory analytic combinations of the utmost importance, which have since been generalized to a notable degree, as for instance, Green's formula.

The fundamental problems of electrostatics belong to the same order of ideas, and it was surely a fine triumph for the theory when the discovery was made of the celebrated theorem on electric phenomena in the interior of hollow conductors, which Faraday rediscovered by experiment without being acquainted with Green's paper. This whole magnificent aggregation became the model for the theories established in mathematical physics, which seem to us to have almost reached perfection, and which exert such a happy influence on the progress of pure analysis by suggesting to it the most beautiful problems.

The theory of functions will afford us another remarkable comparison. The analytic transformations called into play are no different from those already encountered in the steady movement of heat. Certain fundamental problems in the theory of functions of a complex variable have accordingly been able to exchange their abstract enunciation for a physical form, like that of the distribution of the temperature on a closed surface of any connectivity whatever and without radiation, in thermal equilibrium, with two sources of heat necessarily corresponding to equal and opposite flows. Interpreting this geometrically, we find ourselves face to face with a question respecting abelian integrals of the third kind in the theory of algebraic curves.

The preceding examples, in which we have briefly considered only the equations of heat and of attraction, show that the influence of the physical theories is exerted not only on the general nature of the problems to be solved, but even in the details of the analytic transformations. Thus in recent memoirs on partial differential equations, it is customary to denote by the name of Green's formula one which was suggested by the original formula of the English physicist. The theory of dynamic electricity, and that of magnetism, in the hands of Ampère and Gauss were the source of notable advances; the study of curvilinear integrals and of surface integrals owe all their development to this theory, and formulas like that of Stokes, which might also be called Ampère's formula, appeared for the first time in memoirs on physics. The equations of the propagation of electricity, with which are connected the names of Ohm and Kirchhoff, while presenting a strong analogy with those of heat, are often subject to slightly different boundary

conditions; we know how much cable telegraphy owes to the thorough discussion of the integrals of an equation of Fourier's applied to electricity. The equations of hydrodynamics formulated long ago, the equations in the theory of elasticity, those of Maxwell and of Hertz in electro-magnetism, have presented problems analogous to those mentioned above, but under still more varied conditions. Unsurmounted difficulties are met there, it is true, but how many beautiful results are due to the study of particular cases, the number of which we should like to see increased. We should mention also, as of both analytical and physical importance, the profound differences exhibited by propagation according to the phenomena studied. With equations like those of sound there is propagation by waves; with the equation of heat every variation is felt instantly at every distance, but very slightly at very great distances, and therefore it is impossible to speak of a velocity of propagation. In other cases, of which Kirchhoff's equation relative to the propagation of electricity with induction and capacity offers the simplest type, there is a wave-front with a definite velocity, but with a residue behind which is not obliterated. These different circumstances reveal very different properties of the integrals; they have been thoroughly studied only in a small number of particular cases, and these investigations raise problems involving the profoundest ideas in modern analysis.

V.

I shall enter into some analytic details of especial interest to mathematical physics. The question of the generality of the solution of a partial differential equation has presented some apparent paradoxes. For one and the same equation the number of arbitrary functions was not always the same, depending upon the form of the integral considered. Thus Fourier, studying the equation of heat in an unlimited medium, regards it as evident that a solution will be determined if its value is given for $t = 0$, that is to say, as an arbitrary function of the three coördinates, x, y, z; from Cauchy's point of view, on the other hand, it might be considered that the general solution involves two arbitrary functions of three variables. At bottom, the question, in the way in which it was stated for a long time, has no precise significance. In the first place, if only analytic functions are considered, any finite number of functions what-

ever, with any number of independent variables, exhibit from the arithmetical point of view no more generality than a single function of a single variable, since in both cases the totality of the coëfficients in the development forms an enumerable sequence. But there is something more. In fact, besides the conditions expressed by given functions, an integral is subject to conditions of continuity, or must become infinite in a definite way for certain elements; we may then be led to regard the condition of continuity in a given space as equivalent to an arbitrary function, and it is then quite clear in how far the question of the enumeration of the arbitrary functions is badly stated. It is often a delicate matter to prove what conditions determine a solution uniquely, unless one is willing to be satisfied with plausibilities; it is then necessary to state precisely the manner in which the function and certain of its derivatives behave. For instance, in Fourier's problem regarding an indefinite medium, certain hypotheses must be made about the function and its first derivatives at infinity if we wish to show that the solution is unique. Formulas analogous to those of Green are of great service, but the proofs deduced from them are not always absolutely rigorous when we suppose boundary conditions satisfied which are, *a priori* at least, not necessary. This is only another example of the increased demand for rigor in proofs. Let us observe, besides, that the new investigations rendered necessary have often led to a better understanding of the nature of integrals; this is because true rigor is productive, being distinguished in this from another rigor which is purely formal and tiresome, casting a shadow over the problems which it touches.

The difficulties in the proof of the uniqueness of a solution may differ very much according to whether all the integrals of the equations considered are analytic or not. There is an important point here, which shows that it is sometimes necessary to reckon with non-analytic functions, even when we should prefer to avoid them. For example, it cannot be affirmed that Cauchy's problem determines a solution uniquely when the data of the problem are general, that is to say, are not characteristic. This is surely the case when only analytic integrals are considered; but in the case of non-analytic integrals there may be contacts of an infinite order, and in this case the theory is not in advance of the applications; the contrary is true, as is shown by the following example. Does the famous theorem

of Lagrange on the velocity-potential in a perfect fluid hold for a viscous fluid? It has been possible to give examples where the coördinates of the different points of a viscous fluid starting from rest are not expressible as analytic functions of the time starting from the beginning of the movement, and where the rotations, as well as all their derivatives with respect to the time, are zero at that instant, and yet are not identically zero; in this case Lagrange's theorem does not hold. These considerations are sufficient to show the interest there may be in being assured that all the integrals of a system of partial differential equations, continuous and with continuous derivatives up to a definite order in a certain field of real variables, are analytic functions, supposing, of course, that the equations themselves contain only analytic elements. For linear equations there are precise theorems, all the integrals being analytic if the characteristics are imaginary, and very general propositions have also been obtained in other cases. The boundary conditions which we are led to set up are very different according to whether the integrals of the equation in question are analytic or not. A case of the former type is given by the generalized problem of Dirichlet. In this case, an essential part is played by the conditions of continuity, and in general, the solution cannot be extended on the two sides of the continuum with reference to which the data are defined. It is quite different in the second case, where the relation of this continuum to the characteristics is of principal importance, and where the field of existence of the solution is presented under entirely different conditions. All these notions, difficult to state precisely in ordinary language, are fundamental in mathematical physics, and of not less interest in infinitesimal geometry. It will be sufficient to recall the fact that all surfaces of constant positive curvature are analytic, while there are surfaces of constant negative curvature which are non-analytic.

Since the time of the ancients, the vague belief in a certain economy in natural phenomena has made its way; one of the first exact illustrations is furnished by Fermat's principle with regard to the economy of time in the transmission of light. Afterwards it came to be recognized that the general equations of mechanics correspond to a problem of minimum, or more exactly of variation, and in this way there was obtained the principle of virtual velocities, then Hamilton's principle, and the principle of least action. A large number of problems were

thus shown to correspond to the minima of certain definite integrals. This amounted to a very important step in advance, for in many cases the existence of a minimum could be considered self-evident, and in consequence the proof of the existence of a solution was accomplished. This method of reasoning has been of immense service; the greatest geometricians have taken up with it, Gauss in the problem of distribution of an attracting mass corresponding to a given potential, Riemann in his theory of abelian functions. In the present day, our attention has been called to the dangers of this species of demonstration; it is possible that the minima are simply limits that cannot be really attained by actual functions which possess the necessary properties of continuity. We are no longer contented with the plausibilities presented by the method of reasoning which has long been classic. Whether we proceed indirectly, or whether we endeavor to give a rigorous proof of the existence of a function corresponding to the minimum, the road is long and arduous. In other respects, it will be none the less always advantageous to connect a question of mechanics or of mathematical physics with a problem of minimum; in the first place, this affords a fruitful source of analytic transformations, and besides, in the very processes of the investigations of the variations, useful suggestions with regard to the boundary conditions may present themselves; a beautiful example was furnished by Kirchhoff in the delicate investigation of the boundary conditions in the equilibrium of flexion of plates.

VI.

I have been led to dwell particularly on partial differential equations. Examples chosen from rational mechanics and from celestial mechanics would easily show the part played by ordinary differential equations in the progress of these sciences, whose history, as we have seen, has been so closely united with that of analysis. When the hope of integrating by means of simple functions is given up, an effort is made to find developments allowing us to follow a phenomenon as long as possible, or at least to obtain hints as to its qualitative behavior. On the practical side, the methods of approximation form a very important part of mathematics, and thus the most advanced parts of arithmetic are related to the applied sciences. As for series, they occur, to begin with, in the very existence-proofs of the integrals; for instance, Cauchy's first method gives con-

vergent developments as long as the integrals and their differential coefficients are continuous. When any circumstance allows us to predict that this is always the case, we obtain developments always convergent. In the problem of n bodies, we can in this way obtain developments which hold so long as there are no impacts. If the bodies repelled each other instead of attracting, there would be no danger of this happening, and we should obtain developments which would hold indefinitely; unfortunately, however, as Fresnel once said to Laplace, "Nature is not concerned about analytic difficulties," and the heavenly bodies attract instead of repelling each other. We might even be tempted to go further than the great physicist, and say that nature has strewn the path of the analyst with difficulties. Thus, to take another illustration, when a system of differential equations of the first order is given, we can usually decide whether the general solution is stable in the neighborhood of a point or not; and in order to find developments in series which hold for the stable solutions, it is only necessary that certain inequalities be satisfied.

But if we apply these results to the discussion of stability in the equations of dynamics, we find ourselves exactly in the particular unfavorable case. In general, in this very case, it is impossible to make a definite statement with regard to the stability; in the case where a force-function has a maximum, a standard, but indirect, method of reasoning establishes the stability, which cannot be deduced from any development that holds good for every value of the time. Let us not regret these difficulties; they will be the source of future progress. Such also are the difficulties still presented to us, in spite of the great amount of labor expended upon them, by the equations of celestial mechanics; the astronomers have deduced from them, since the time of Newton, by means of practically convergent series and of happily conducted approximations, pretty nearly all that is necessary for the prediction of the movements of the celestial bodies. The analysts would demand more, but they hardly hope any longer to perform the integration by means of simple functions or of developments which are always convergent. What has been best learned from some recent remarkable investigations, is the immense difficulty of the problem; a new road has been opened, however, by the study of particular solutions, like periodic solutions and asymptotic solutions, which have already been put into use. It is perhaps not so

much for the sake of practical necessities as for the sake of not admitting defeat, that analysis will never reconcile itself to abandoning without a positive victory a subject which has been the scene of so many brilliant triumphs; and, besides, what finer field could be found to test the strength of the newly born or rejuvenated theories of the modern doctrine of functions than this classic problem of n bodies?

The analyst is rejoiced to meet in the applications equations which he can integrate with known functions, with transcendentals already classified. Such meetings are unfortunately rare; the pendulum problem, the classic cases of the motion of a solid body about a fixed point, are examples where the integration can be performed by the use of elliptic functions. It would also be extremely interesting to meet a question in mechanics which could be the origin of an important discovery in the theory of functions, such as the discovery of a new transcendental enjoying some remarkable property; I should be at a loss to give an illustration, unless I were to refer the origin of the theory of elliptic functions to the pendulum. The relations between theory and application are here much less intimate than in the questions of mathematical physics considered above. This explains the fact that, during the last forty years, the study of the ordinary differential equations connected with analytic functions has had for the most part a totally abstract theoretical character. The pure theory has been notably in the lead; we have had occasion to say that it was good that it was so, but here there is evidently a question of degree, and one could wish to see some old problem profit by the progress that has been made. We should not be especially at a loss to give some examples in this connection, and I will call to mind only those linear differential equations involving arbitrary parameters whose singular values are roots of transcendental integral functions, and that one in particular which establishes a correspondence between the successive harmonics of a vibrating membrane and the poles of a meromorphic function.

It occurs also that the theory may be an element of classification by leading to the investigation of the conditions under which the solution belongs to a definite type, as for instance, when the integral is a uniform function. There have been and there will be again plenty of interesting discoveries in this field. The case of the movement of a heavy solid body, treated by Mme. de Kowalewski, where abelian functions were employed is a remarkable instance.

VII.

In studying the mutual relations between analysis and mechanics and mathematical physics, we have more than once found the road opened by infinitesimal geometry, which has proposed so many celebrated problems; in many difficult questions, a happy combination of the calculus with synthetic methods has been the means of great advances, as is shown by the theories of applicable surfaces and of triply orthogonal systems. There is another branch of geometry which is of great importance in some analytical investigations; I mean the geometry of situation, or analysis situs. We know that Riemann made from this point of view a complete study of the continuum of two dimensions, upon which rests his theory of algebraic functions and their integrals. When this number of dimensions is increased, the questions of analysis situs become necessarily more complicated; geometric intuition ceases, the study becomes purely analytic, and the thought is guided only by analogies which may be deceptive and require close investigation. The theory of algebraic functions of two variables, which transports us into a space of four dimensions, without reaping so much advantage from analysis situs as does the theory of functions of one variable, owes to it, nevertheless, some useful orientations. Here is still another class of questions which involve the geometry of situation; in the study of curves drawn on a surface and defined by differential equations, the connectivity of this surface plays an important part; this is what occurs in the case of geodesic lines. The notion of connectivity, moreover, presented itself long ago in analysis, when the study of electric currents and of magnetism had led to non-uniform potentials; in a more general manner, certain multiform integrals of some partial differential equations are met in difficult theories like that of diffraction, and there are various investigations to be pursued in this direction.

From a different point of view, I ought to mention also the relations between algebraic analysis and geometry which appear so elegantly in the theory of groups of finite order. A regular polyhedron, an icosahedron for example, is on the one hand the well known solid; it is also for the analyst a group of finite order corresponding to the different ways of bringing the polyhedron into coincidence with itself. The investigation of all the types of groups of movements of finite order interests not

only geometricians, but crystallographers as well; it amounts essentially to a study of the groups of ternary linear substitutions of determinant ± 1, and leads to the thirty-two classes of symmetry for the molecular structure of crystals. The corresponding groupings of polyhedra in systems so as to fill up space exhaust all the possibilities in the investigation of the structure of crystals. Since the epoch when the notion of groups was introduced into algebra by Galois, it has undergone in different ways remarkable developments, to such an extent that it is met to-day in every branch of mathematics. In the applications it appears to us above all as an admirable instrument of classification. Whether we consider substitution groups or Sophus Lie's transformation groups, whether we consider algebraic equations or differential equations, this comprehensive theory permits us to take account of the degree of difficulty of the problems treated and teaches us to employ the special circumstances which present themselves in each; for this reason it must prove as essential to mechanics and to mathematical physics as it is to pure analysis.

Mechanics and mathematical physics have been developed to the point where a mathematical form may be given to almost all their theories; certain hypotheses and the knowledge of elementary laws have led to the differential relations which constitute, at least for the time, the final form of statement of these theories. Little by little they have seen their field enlarged by the principles of thermodynamics; and to-day chemistry is tending, in its turn, to take on a mathematical form. I shall call to witness only the celebrated memoir by Gibbs on the equilibrium of chemical systems, of a character so analytic that it requires some effort for chemists to recognize laws of great importance under their algebraic cloak.

It seems that chemistry has now emerged from the pre-mathematical period in which every science takes its start, and that a day must come when vast theories will be established, analogous to those of our present mathematical physics, but of a much wider scope, and including all the physico-chemical phenomena. It would be premature to inquire whether analysis will find in their developments the source of new advances; we do not even know what analytic types we may encounter. I have constantly spoken of differential equations governing phenomena; will this be always the final form into which a theory is crystallized? I know nothing about it with certainty,

but yet we must remember that several hypotheses of a more or less experimental character have been made; among them is one which may be called the principle of *non-heredity*, which postulates that the future of a system depends only on its present state and on its state at the infinitely near instant, or more briefly, that accelerations depend only on positions and velocities. We know that in certain cases this hypothesis is not admissible, at least with the quantities directly considered; in this connection a misuse has sometimes been made of the memory of matter, as if it remembered its past, and the life of a piece of steel has been spoken of in moving terms. Various attempts have been made to present a theory of these phenomena where a far-reaching past seems to intervene; this is not the place to discuss them. An analyst may think that in cases of such complexity it will be necessary to abandon the form of differential equations and to be resigned to the consideration of *functional equations* involving definite integrals which will be the evidence of a sort of heredity. In view of the interest attaching at the present time to functional equations, one might believe in a presentiment of future needs.

VIII.

After speaking of non-heredity, I hardly dare to touch on the question of the applications of analysis to biology. There is no doubt that functional equations of biological phenomena will not be constructed so easily;* the attempts made hitherto have been confined to a very modest range of ideas. However, the effort is making to come out of a purely qualitative field and to introduce quantitative measurements. In the question of the variation of certain characters, reliance is placed on processes of measurement and on statistical data which are interpreted by curves of frequency. The modifications of these curves with successive generations and their decomposition into distinct curves could give the measure of stability of species or of the rapidity of mutations, and we know what interest attaches to these questions in recent botanical investigations. In all this there is such a large number of parameters, that the question is whether the infinitesimal method itself can be of any service.

*In his article on "Lamarck's principle and the heredity of bodily changes," M. Giard says of heredity: "It is an integral, it is the sum of variations produced in each foregoing generation by the primary factors of evolution." See Controverses transformistes, p. 135.

Some laws of a simple arithmetical character, like those of Mendeleëff, have sometimes renewed confidence in the old aphorism which I quoted at the start, that all things are explained by numbers; but in spite of justifiable hopes, it is clear that, taken as a whole, biology is still far from entering into a really mathematical period.

It is otherwise with political economy, according to certain economists. Following Cournot, the school of Lausanne has made an extremely interesting effort to introduce mathematical analysis into political economy. Under certain hypotheses which agree at least with certain limiting cases, there is found in learned treatises an equation between quantities of merchandise and their prices which recalls the equation of virtual velocities in mechanics; this is the equation of economic equilibrium. In this theory, a function of the quantities plays an essential part not unlike that of the potential function. Moreover, the leading representatives of the school insist on the analogy between economic and mechanical phenomena.

"As rational mechanics" says one of them, "considers material points, so pure economics considers the homo œconomicus." Here also, of course, are found again equations analogous to those of Lagrange, the inevitable pattern of all mechanics. With all admiration for these daring performances, we are inclined to fear that the authors have neglected certain concealed masses, as Helmholtz and Hertz would have said. But whatever happens to them, these theories contain a curious application of mathematics, which has already been of great service, at least in some well circumscribed cases.*

Gentlemen, I have finished this brief history of some of the applications of analysis with the reflections which it has suggested to me from time to time. I have given a résumé far from complete; for instance, I have omitted to speak of the calculus of probabilities, which demands such subtlety of thought, and whose artifices Pascal refused to explain to the Chevalier de Méré because he was not a geometrician. Its practical utility is of the first order, its theoretical interest has always been great, and has been still more increased recently, thanks to the investigations which Maxwell called statistical and which tend to exhibit mechanics in an entirely new light.†

*See upon this topic: La méthode mathématique en économie politique, by E. Bonoier, and Petite traité d'économie politique mathématique, by H. Laurent.

†See Elementary principles in statistical mechanics, by J. W. Gibbs, and Léçons sur la theorie des gaz, by L. Boltzmann.

I hope, however, to have shown in this sketch the origin and reason of the close bonds which unite analysis with geometry and with physics, more generally with every science bearing on quantities which can be measured numerically. The mutual influence of analysis and the physical theories has been in this respect particularly instructive. What has the future in reserve? More difficult problems, corresponding to a higher order of approximation, will introduce complications that we can only vaguely foresee, speaking, as I did just now, of functional equations replacing systematically the differential equations of our present time, or again of integration of equations infinite in number, and involving an infinity of unknown functions. But whatever happens, mathematical analysis will always remain that language which, in Fourier's words, "has no symbols to express confused thoughts," a language endowed with a wonderful power of transformation and able to condense within its formulas an immense number of results.

ON THE CLASS OF THE SUBSTITUTIONS OF VARIOUS LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

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1. In a recent memoir * by M. Edmond Maillet the question of the possible number of real elements of a geometric configuration (such as the 27 straight lines on a cubic surface) is made to depend upon the class of the substitutions of the Galois group G of the equation determining the elements or of any known group containing G. In view of such an application in various geometric and function-theoretic problems, Maillet emphasizes the importance of a knowledge of the class of the substitutions of various linear modular groups. For the general linear group on m variables with coefficients modulo n, Maillet determines completely the class of its substitutions when n is a prime, while for n a power of a prime he determines a set of

* *Annales de l'Université de Toulouse* (2), vol. 6 (1904), pp. 277–349. In a paper to appear in the July number of the *Annals of Mathematics*, I obtain wide generalizations of Maillet's geometric results, the methods employed being much simpler than his.

numbers among which the classes must occur. The analysis employed to obtain these incomplete results is very elaborate.

Some years ago I effected an immediate determination of the class of the substitutions S of the m-ary linear homogeneous group in the Galois field of order p^n, when represented * as a substitution group on the $p^{nm}-1$ symbols $\lambda_1\xi_1+\cdots+\lambda_m\xi_m$, the λ's being arbitrary marks not all zero of the $GF[p^n]$ and the ξ's being arbitrary variables. If S leaves fixed the linear functions $f_1, \cdots, f_k$ of the ξ's, it leaves fixed $\mu_1 f_1+\cdots+\mu_k f_k$, for arbitrary marks μ_i. Any linear homogeneous function with coefficients in the $GF[p^n]$ is either linearly independent of $f_1, \cdots, f_k$ or else is expressible in the form $\sum\mu_i f_i$. Hence the number of distinct linear functions whose coefficients are marks not all zero and which are invariant under S is of the form $p^{nt}-1$. Hence the class (the number of letters displaced) is $p^{nm}-p^{nt}$ ($t=0, 1, \cdots,$ or $m-1$), the identity substitution not being considered. That there exist substitutions of each of these classes follows † as in § 2.

Although the preceding representation of the linear group is the more advantageous in the theory of linear groups and renders the preceding theorem quite obvious, I shall employ in the remainder of the paper the representation ‡ on the letters $l_{\xi_1,\ldots,\xi_m}$, written also $(\xi_1, \cdots, \xi_m)$, the ξ's being marks of a Galois field (or else integers modulo n), this method (used also by Maillet) being more suitable for the applications.

2. THEOREM I. *The general m-ary linear homogeneous group in the $GF[p^n]$ contains substitutions of classes $p^{nm}-p^{nt}$ ($t=0, 1, \cdots, m-1$) exclusively, and is m-fold incompletely transitive. For the non-homogeneous group the only additional class is p^{nm}; it is $(m+1)$-fold incompletely transitive. The class of the two groups is $p^{nm}-p^{n(m-1)}$.*

As the substitution $\xi_i'=\xi_i+1 (i=1, \cdots, m)$ displaces all p^{nm} letters, and as the non-homogeneous group is transitive, it

* Linear Groups, § 98; *American Journal*, vol. 23 (1901), pp. 337–377.

† By taking the transposed of the matrices of the substitutions there given.

‡ To connect the two methods, we note that S replaces $l_{\xi_1,\ldots,\xi_m}$ by $l_{\xi_1',\ldots,\xi_m'}$, where $\xi_i'=\sum_{j=1}^{m} a_{ij}\xi_j$, and replaces $\Sigma\lambda_i\xi_i$ by $\Sigma\lambda_i'\xi_i$, where $\lambda_i'=\sum_{j=1}^{m} a_{ji}\lambda_j$. It follows readily that S leaves fixed the same number of symbols in each case and that the corresponding permutations are *similar*. Further, for the general linear group, the hyperorthogonal, the abelian, the two hypoabelian, and the first orthogonal groups, the transposed matrix belongs to the group, so that the two representations are identical.

suffices to determine the class of the substitutions leaving fixed $l_{0,\cdots,0}$, viz., of the homogeneous substitutions

$$\xi_i' = \sum_{j=1}^{m} \alpha_{ij}\xi_j \qquad (i = 1, \cdots, m). \tag{1}$$

But if (1) leaves fixed the letters $(\xi_i) \equiv (\xi_1, \cdots, \xi_m)$ and (ξ_i^*), so that

$$\xi_i = \sum \alpha_{ij}\xi_j, \qquad \xi_i^* = \sum \alpha_{ij}\xi_j^* \qquad (i = 1, \cdots, m),$$

it leaves fixed $(\alpha\xi_1 + \beta\xi_1^*, \cdots, \alpha\xi_m + \beta\xi_m^*)$, for arbitrary marks α, β. Hence all the letters left fixed by (1) may be expressed in the form

$$\left(\sum_{j=1}^{t} \mu_j \xi_1^{(j)}, \cdots, \sum_{j=1}^{t} \mu_j \xi_m^{(j)}\right), \ (\mu_1, \cdots, \mu_t \text{ arbitrary marks}), \tag{2}$$

in which the letters $(\xi_i^{(1)}), \cdots, (\xi_i^{(t)})$ are independent in the sense that (2) is $(0, \cdots, 0)$ if and only if $\mu_1 = \cdots = \mu_t = 0$. Hence the number of fixed letters is p^{nt} and the classes $p^{nm} - p^{nt}$.

It remains to exhibit a substitution which leaves fixed only the p^{nt} letters (ξ_i) with $\xi_1, \cdots, \xi_t$ arbitrary, $\xi_{t+1} = \cdots = \xi_m = 0$. If $p^n > 2$, we take

$$\xi_i' = \xi_i \ (i = 1, \cdots, t), \ \xi_j' = \alpha\xi_j \ (j = t+1, \cdots, m), \ \alpha \neq 0, 1.$$

If $p^n = 2$, $m = 1$, S is the identity. If $p^n = 2$, $m > 1$, we take

$$\xi_i' = \xi_i (i = 1, \cdots, t-1), \ \xi_m' = \xi_m, \ \xi_j' = \xi_j + \xi_{j+1} (j = t, \cdots, m-1);$$

$$\xi_i' = \xi_{i+1} (i = 1, \cdots, m-1), \qquad \xi_m' = \xi_m' + \xi_1;$$

according as $t > 0$ or $t = 0$.

Confining our attention to substitutions of determinant 1, we derive the

COROLLARY. *The special m-ary linear homogeneous group in the $GF[p^n]$ contains substitutions of classes $p^{nm} - p^{nt}$ $(t = 0, 1, \cdots, m-1)$ exclusively and is $(m-1)$-fold incompletely transitive. The special non-homogeneous group contains in addition only substitutions of class p^{nm} and is m-fold incompletely transitive.*

3. THEOREM II. *For any integers m and n, the general m-ary linear homogeneous group modulo n contains substitutions of classes*

$n^m - d_i$ exclusively, where d_i ranges over the divisors of n^m. For the non-homogeneous group the only additional class is n^m.*

The proof begins as in § 2. The letters invariant under (1) may be expressed in the form (2), which now, however, may reduce to $(0, \cdots, 0)$ without necessitating $\mu_1 \equiv \cdots \equiv \mu_t \equiv 0 \pmod{n}$. The number of distinct letters (2) is the number d of the distinct linear functions (compare § 1) $\sum_{j=1}^{t} \mu_j f_j$, when $\mu_1, \cdots, \mu_t$ range independently over all integers modulo n, where

$$f_j = \xi_1^{(j)}\xi_1 + \xi_2^{(j)}\xi_2 + \cdots + \xi_m^{(j)}\xi_m \quad (\xi_1, \cdots, \xi_m \text{ arbitrary variables}).$$

Hence d is the number of distinct elements of the *modul*† $[f_1, \cdots, f_t]$. It is thus readily proved that d is a divisor of n^m. This result may also be shown‡ by a direct consideration of the number of incongruent sets of solutions of

$$(1') \quad a_{ii}\xi_1 + \cdots + a_{i\,i-1}\xi_{i-1} + (a_{ii} - 1)\xi_i + a_{i\,i+1}\xi_{i+1} + \cdots + a_{im}\xi_m \equiv 0 \pmod{n}.$$

It remains to exhibit a substitution which leaves fixed exactly d_i letters. In view of § 2 we may assume here that n is not prime. For $n > 2$, it suffices to employ a combination of the following types of substitutions:

$$\xi_1' = (1 + \delta)\xi_1, \quad \xi_i' = \xi_i (i > 1), \quad \delta \text{ a divisor} < n \text{ of } n;$$

$$\xi_1' = \xi_1 + \delta_2\xi_2, \quad \xi_2' = \xi_2 + \delta_1\xi_1, \quad \xi_i' = \xi_i (i > 2), \quad \delta_1, \delta_2 \text{ divisors of } n,\ \delta_1\delta_2 \neq 1;$$

$$\xi_1' = 2\xi_1 + \xi_2, \quad \xi_2' = \xi_1 + \xi_2, \quad \xi_i' = \xi_i (i > 2);$$

which leave fixed only the letters $(\xi_1, \cdots, \xi_m)$ for which, respectively,

$$\xi_1 \equiv 0 \pmod{n/\delta}; \quad \xi_1 \equiv 0 \pmod{n/\delta_1},\ \xi_2 \equiv 0 \pmod{n/\delta_2};$$
$$\xi_1 \equiv \xi_2 \equiv 0 \pmod{n}.$$

* Positive integral divisors less than n^m.

† By an evident extension of Dedekind's definition, Zahlentheorie (4), p. 493, we may reduce the integral coefficients modulo n.

‡ Bachmann, Arithmetik der quadratischen Formen (1), p. 353. The final step in the proof may also be derived from this reference.

4. THEOREM III. *The general or special $2m$-ary linear abelian group in the $GF[p^n]$ contains substitutions of classes $p^{2nm} - p^{nt}(t = 0, 1, \cdots, 2m-1)$ exclusively.*

In view of Theorem I, it remains only to show that these classes actually occur. Suppressing unaltered variables, we set

$$L_{ii}: \xi_i' = \xi_i + \eta_i; \quad U_i: \xi_i' = \xi_i + \eta_i,\ \eta_i' = -\xi_i.$$

Then $U_1U_2\cdots U_r$ $(r \leqq m)$ leaves fixed only the letters with $\xi_i = \eta_i = 0$ $(i = 1, \cdots, r)$; while $U_1U_2\cdots U_rL_m (r < m)$ leaves fixed the letters with $\xi_i = \eta_i = 0$ $(i = 1, \cdots, r)$, $\eta_m = 0$.

5. THEOREM IV. *The m-ary group H of all hyperorthogonal* substitutions in the $GF[p^{2s}]$ contains substitutions of classes $p^{2sm} - p^{2st}(t = 0, 1, \cdots, m-1)$ exclusively. The same is true of the subgroup H_1 of determinant unity.*

We apply Theorem I. For H we use a product of the substitutions

$$T_{i,\tau}: \xi_i' = \tau\xi_i,\ \xi_j' = \xi_j\,(j \neq i), \qquad (\tau^{p^s+1} = 1). \tag{3}$$

For H_1 with $p > 2$, the hyperorthogonal substitution on ξ_1, ξ_2,

$$V_{12} \equiv \begin{pmatrix} 1 + \rho^{\frac{1}{2}(p^s+1)} & \rho \\ -\rho^{p^s} & 1 - \rho^{\frac{1}{2}(p^s+1)} \end{pmatrix}, \tag{4}$$

$$\rho = \text{prim. root } GF[p^{2s}],$$

leaves fixed only the letters with $\xi_2 = -\rho^{\frac{1}{2}(p^s-1)}\xi_1$. Now $W \equiv T_{3,\tau}T_{4,\tau^{p^s}}\cdots T_{2t+1,\tau}T_{2t+2,\tau^{p^s}}$ for $\tau^{p^s+1} = 1, \tau \neq 1$, leaves fixed exactly $p^{2s(m-2t)}$ letters; $V_{12}W$ exactly $p^{2s(m-2t-1)}$ letters. If $m = 3$, a substitution displacing all the letters is $T_{1,\tau}T_{2,\tau}T_{3,\tau^{2p^s}}$, $\tau^{p^s+1} = 1, \tau^2 \neq 1$. If $p = 2$, we use $\left(\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right)$ instead of V_{12}.

COROLLARY. The theorem holds for the hyperabelian† group.

6. THEOREM V. *The group O of all m-ary orthogonal substitutions in the $GF[p^n]$ contains substitutions of classes $p^{nm} - p^{nt}$ $(t = 0, 1, \cdots, m-1)$ exclusively. The subgroup O_1 of orthogonal substitutions of determinant $+1$ contains substitutions of classes $p^{nm} - p^{2kn}$ $(k = 0, 1, \cdots, \frac{1}{2}m - 1)$ exclusively if m is even, but of classes $p^{nm} - p^{(2k-1)n}$ $(k = 1, \cdots, \frac{1}{2}(m-1))$ exclusively if m is odd.*

* Leaving invariant $\Sigma\xi_i^{p^s+1}$, Linear Groups, p. 131.

† Linear Groups, p. 115, p. 116 (§138).

For O the result follows from Theorem I and the existence of the orthogonal substitution $C_1C_2\cdots C_t$, where C_i changes the sign of ξ_i but leaves the remaining variables unaltered.

For O_1, we show that if (1) is orthogonal of determinant $+1$, the number of sets of solutions in the $GF[p^n]$ of (1′) is of the form p^{2kn} for m even, and $p^{(2k-1)n}$ for m odd. To this end we apply the theorem * that in the determinant

$$(5) \qquad \begin{vmatrix} a_{11}-1 & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22}-1 & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

of equations (1′) if all the minors of order $2k$ vanish, then all the minors of order $2k-1$ vanish. Further, (5) vanishes if m is odd, as shown by the fact that the characteristic equation of a substitution of O_1 is a reciprocal equation $-\rho^n+\cdots+1=0$. Finally, $C_1C_2C_3\cdots C_{2t}$ occurs in O_1.

7. Theorem V holds true for the second orthogonal group O' defined by the invariant $\xi_1^2+\cdots+\xi_{m-1}^2+\nu\xi_m^2$, ν a particular not-square in the $GF[p^n]$. Indeed by the transformation $\xi_m^*=\nu^{\frac{1}{2}}\xi_m$, belonging to the $GF[p^{2n}]$, O' becomes O; while any minor in the matrix (5) obtained from a substitution of O' equals a constant times the corresponding minor for O. Hence, by Linear Groups, pages 156–158, *Theorem V holds true for the group of m-ary linear homogeneous substitutions which leave invariant a quadratic form* $\Sigma\gamma_{ij}\xi_i\xi_j$ *with coefficients in the* $GF[p^n]$, $p>2$, *and of discriminant* $\neq 0$.

The group defined by a quadratic form in the $GF[2^n]$ is either simply isomorphic with the abelian group or else is conjugate with one of the hypoabelian groups G_λ, $\lambda=0$ or λ' (Linear Groups, pages 197–201).

8. THEOREM VI. *The* $2m$*-ary hypoabelian group* G_λ *in the* $GF[2^n]$ *contains substitutions of classes* $2^{2mn}-2^{nt}$ $(t=0, 1, \cdots, 2m-1)$ *exclusively.*†

We first apply Theorem I. Next, we note that for G_0 the letters left fixed by M_1, M_1M_2, M_1N_{121}, $M_1M_2N_{121}$, $M_1M_2N_{121}M_3$,

* I derived this theorem independently but later found it stated explicitly by Taber, *Proc. Lond. Math. Soc.*, vol. 27 (1896), p. 613, and inadequately by Voss, *Math. Ann.*, vol. 13, p. 330.

† The case $\lambda=0$, $m=1$, $n=1$ is to be excluded as G_λ is then the identity.

$M_1M_2N_{121}P_{13}$ are given by respectively 1, 2, 3, 4, 5, 6 independent linear relations between the variables, so that a suitable combination of these types leads to a substitution of class $2^{2nm} - 2^{nl}$. For $G_{\lambda'}$, we note that L and $M_1M_mN_{m11}$ (in the notations of Linear Groups, page 201) leave fixed the letters with $\xi_m = \eta_m = 0$, $\xi_m = \eta_m = \xi_1 = \eta_1 = 0$, respectively.

9. THEOREM VII. *The subgroup* * J_λ *of index two under* G_λ *contains substitutions of classes* $2^{2nm} - 2^{2nj} (j = 0, 1, \cdots, m-1)$ *exclusively.*

I have established this theorem for low values of m with n arbitrary; but as the method seems unwieldy for general m, I suppress the proof. This remarkable theorem in connection with Theorem VI furnishes an analogue to Theorem V.

THE UNIVERSITY OF CHICAGO,
February 10, 1905.

NOTE ON A PROBLEM IN MECHANICS.

BY MR. A. M. HILTEBEITEL.

(Read before the American Mathematical Society, February 25, 1905.)

IN an article† inserted in the twenty-eighth volume of the *Giornale di Matematiche*,‡ on the separation of the variables in the equations of the motion of a body acted upon by two fixed centers of force, the author, Dr. Carlo Bonacini, states with inadequate proof that the separation of the variables in the equations of motion is possible only when the two forces vary inversely as the squares of the respective distances of the body from the fixed centers.

That the variables can be separated when the forces vary inversely as the squares of the distances has been known since

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* Linear Groups, p. 206, § 205.

† "Sulla separazione delle variabili nelle equazioni del moto di un punto soggetto all'azione di due centri fissi," *Giornale di Matematiche*, vol. 28 (1890), pp. 132–137. (Date of article, May, 1889.)

‡ The object of the paper is given by the author in the following words: "Noi dimostreremo a questo proposito che la separazione delle variabili nelle equazioni del moto è possibile solo quando le due forze sono Newtoniane."

the classic memoirs of Euler,* but it is not true that this is the only law of force for which separation of the variables is possible. To convince one's self of the latter fact it is only necessary either to refer to Lagrange's second memoir † on this problem, where it is shown that the variables are separable when the law of force is a certain linear combination of the inverse square and the direct distance,‡ or to remark that in the case of forces varying directly as the distances the variables are separated in the differential equations of the motion, expressed in ordinary rectangular coördinates.

It is the object of the present note to point out the error in Dr. Bonacini's reasoning and to show that Lagrange's form of the forces just referred to is the most general case admitting of the separation of the variables, when the forces are functions only of the respective distances.

Let K_1, K_2, M represent the two fixed centers and the moving point respectively. Let r_1 and r_2 denote the distances K_1M and K_2M respectively. Finally, let $U(r_1)$ and $V(r_2)$ denote the forces acting upon M and constantly directed toward the centers K_1 and K_2 respectively. Lagrange showed in the above mentioned memoir that the variables can be separated, whether

*Euler: "De motu corporis ad duo virium centra attracti," *St. Petersburg Memoirs* for 1764, published 1766, vol. 10, pp. 207–242.

"Un corps étant attiré en raison réciproque quarrée des distances vers deux points fixes donnés, trouver les cas où la courbe décrite par ce corps sera algébrique," *Berlin Memoirs* for 1760, published 1767, pp. 228–249.

"De motu corporis ad duo centra virium fixa attracti," *St. Petersburg Memoirs* for 1765, published 1767, vol. 11, pp. 152–184.

† Lagrange: "Recherches sur le mouvement d'un corps qui est attiré vers deux centres fixes," *Turin Memoirs* for 1766–69, vol. 4, pp. 215–243; Œuvres vol. 2 (1868), pp. 94–121.

‡ Among other memoirs appearing before 1889 (the date of Bonacini's paper) which make mention of the fact that the variables are separable when the forces are of this nature, are the following:

Liouville: "Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer." *Journ. de Mathématiques*, vol. 11 (1846), pp. 345–378; vol. 12 (1847), pp. 410–444; note vol. 13 (1848), pp. 34–37.

Serret: "Mouvement d'un point materiel attiré par deux centres fixes, en raison inverse du carré des distances," *Journ. de Mathématiques*, vol. 13 (1848), pp. 13–34.

Desboves: "Sur le mouvement d'un point matériel attiré en raison inverse du carré des distances par deux centres mobiles," *Journ. de Mathématiques*, vol. 13 (1848), pp. 369–396.

Bertrand: "Mémoire sur quelque-unes des formes les plus simples que puissent présenter les intégrales des équations différentielles d'un point matériel," *Journ. de Mathématiques*, ser. 2, vol. 2 (1857), pp. 113–140.

Jacobi: "Anziehung eines Punktes nach zwei festen Centren," Vorlesungen über Dynamik, No. 29 (read 1842–43), 1866; Gesammelte Werke, Supplement (1884), pp. 221–231.

the motion takes place in a plane or in space when the forces are of the form

$$(1) \qquad U = 2cr_1 + \frac{c_1}{r_1^2}, \qquad V = 2cr_2 + \frac{c_2}{r_2^2},$$

where c, c_1, c_2, are any constants. We shall show that these are the most general forms of the functions U and V for which the variables are separable. To this end we shall make use of results established by Liouville and Morera and employed by Dr. Bonacini in his argument.

Liouville, in the memoirs mentioned above, proved that separation of the variables is possible whenever the linear element and the potential function have the forms

$$(2) \qquad ds^2 = \lambda(m d\mu^2 + n d\nu^2), \quad P = [f(\mu) + F(\nu)]/\lambda$$

respectively, where m and $f(\mu)$ are any functions of μ, n and $F(\nu)$ any functions of ν, and λ is a function of μ and ν in which the variables μ and ν are separated. He showed further that the linear element will take this form only for elliptic coördinates and their limiting forms, namely, cartesian, parabolic, and polar coördinates.

Later Morera* showed that the conditions (2) are also necessary for the separation of the variables in the equations of the motion of a point in a plane.

Now, retaining the method and notation of Dr. Bonacini's paper, let σ and δ be the plane elliptic coördinates and a the distance between the foci. Then the linear element takes the form

$$(3) \qquad ds^2 = \frac{\sigma^2 - \delta^2}{4}\left(\frac{d\sigma^2}{\sigma^2 - a^2} - \frac{d\delta^2}{\delta^2 - a^2}\right),$$

and the potential function must have the form

$$(4) \qquad P = 4\frac{f(\sigma) + F(\delta)}{\sigma^2 - \delta^2} \equiv \frac{f(r_1 + r_2) + F(r_1 - r_2)}{r_1 r_2},$$

and must evidently satisfy the partial differential equation

$$(5) \qquad \frac{\partial^2 r_1 r_2 P}{\partial r_1^2} = \frac{\partial^2 r_1 r_2 P}{\partial r_2^2}.$$

* "Sulla separazione delle variabili nelle equazioni del moto di un punto materiale su una superficie," *Atti della R. Accad. di Torino*, vol. 16 (1880-81), pp. 276-295.

Assuming the motion in the problem of two fixed centers to take place in a fixed plane passing through the line of centers, the two fixed centers themselves being taken as the foci of the system of elliptic coördinates, then it is required to find what the forms of U and V must be in order that the function

$$P = U(r_1) + V(r_2) \tag{6}$$

may satisfy the equation (5).

From (5) and (6) we have

$$\frac{\partial^2\{r_1 r_2[U(r_1) + V(r_2)]\}}{\partial r_1^2} = \frac{\partial^2\{r_1 r_2[U(r_1) + V(r_2)]\}}{\partial r_2^2},$$

which may be written as follows:

$$\frac{d^2U}{dr_1^2} + \frac{2}{r_1}\frac{dU}{dr_1} = \frac{d^2V}{dr_2^2} + \frac{2}{r_2}\frac{dV}{dr_2}. \tag{7}$$

The first member of this equation is a function of r_1 alone and the second of r_2 alone. Hence we may set each member equal to the same constant and integrate each equation so obtained.

At this point Dr. Bonacini says: "In questa relazione il primo membro è funzione della sola r_1 ed il secondo della sola r_2, bisognerà quindi che l'uno e l'altro siano uguali a zero." By setting each member equal to zero and integrating, he was led to his erroneous conclusion.

Setting each member equal to the same constant $6c$, we have

$$\frac{d^2U}{dr_1^2} + \frac{2}{r_1}\frac{dU}{dr_1} = 6c, \tag{8}$$

$$\frac{d^2V}{dr_2^2} + \frac{2}{r_2}\frac{dV}{dr_2} = 6c. \tag{9}$$

Since these equations have exactly the same form, it is sufficient to consider one of them, say the first.

If we set $z = \log r_1$, or $r_1 = e^z$, the equation becomes

$$\frac{d^2U}{dz^2} + \frac{dU}{dz} = 6c \cdot e^{2z},$$

a linear differential equation with constant coefficients, of which the complete integral is

$$U = c_4 + c_1 e^{-z} + c e^{2z},$$

c_1 and c_4 being constants of integration. Hence the complete integral of equation (8) is

$$U = c_4 + \frac{c_1}{r_1} + c r_1^2,$$

and similarly the complete integral of equation (9) is

$$V = c_5 + \frac{c_2}{r_2} + c r_2^2,$$

where c_2, c_5 are constants of integration. And therefore the potential function P is

$$P = c r_1^2 + c r_2^2 + \frac{c_1}{r_1} + \frac{c_2}{r_2} + c_3, \tag{10}$$

which shows that the forces directed toward the two fixed centers are of the form first introduced by Lagrange, when the motion takes place in a fixed plane passing through the fixed centers of force.

To extend this to the general case of motion in space it is only necessary to remark that the linear element has the same form as before and the potential function differs only by the additional term

$$-\tfrac{1}{2}\frac{a^2}{r^2},$$

where r denotes the distance of the moving point from the line of centers and a is the constant of areas described by r as it rotates about the line of centers. In elliptic coordinates this term becomes

$$-\tfrac{1}{2}\frac{a^2}{r^2} = \frac{a^2 a^2}{\sigma^2 - \delta^2}\left\{\frac{1}{\sigma^2 - a^2} - \frac{1}{\delta^2 - a^2}\right\},$$

which is itself of the required form. Hence the potential function for the general case can be put into the form (4), which form is necessary and sufficient for the separation of the variables in the equations of motion.

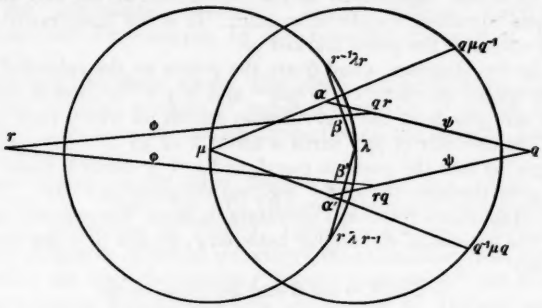
PRINCETON, NEW JERSEY,
February 14, 1905.

A GEOMETRIC CONSTRUCTION FOR QUATERNION PRODUCTS.

BY PROFESSOR IRVING STRINGHAM.

(Read before the San Francisco Section of the American Mathematical Society, February 25, 1905.)

In the construction here given the lines actually inserted in the figure are supposed to lie on a hypersphere (spherical space of three dimensions) having a radius equal to the linear unit, and in particular all of the lines except the four marked ϕ, ψ, lie on a spherical surface passing through the extremities of the quaternion unit vectors i, j, k. The Greek letters that indicate the position of points are vectors drawn from the centre of the sphere to its surface and q, r, qr, rq are quaternion directors to points on the hypersphere. The centers of the sphere and hypersphere are coincident and are at the origin of vectors and of quaternion directors. The two circles of the figure have their centers at λ and μ.



The quaternions whose products are to be constructed are given in the form

$$q = \cos\phi + \lambda \sin\phi,$$
$$r = \cos\psi + \mu \sin\psi \qquad [Tq = Tr = 1].$$

It is well known that with a pair of quaternions q, r, there is associated a definite plane, whose equation is

$$\lambda t - tr^{-1}\lambda r = 0,*$$

* *Transactions Amer. Math. Soc.*, vol. 2, p. 194.

in which r, through progressive multiplication by q, is turned about the origin into the position qr.* It is easily verified that r, qr and $\lambda + r^{-1}\lambda r$ satisfy the above equation for all values of ϕ and ψ. A plane in which qr must lie is therefore completely determined by the lines r and $\lambda + r^{-1}\lambda r$, while $r^{-1}\lambda r$ is known to be that vector which is got by rotating λ about μ as an axis through the angle 2ψ.†

Similarly, another plane, in which qr must lie, has for its equation

$$q\mu q^{-1}t - t\mu = 0,$$

which is satisfied by q, qr and $\mu + q\mu q^{-1}$, and is therefore completely determined by the lines q and $\mu + q\mu q^{-1}$. The two planes meet in a straight line, for

$$S\lambda q\mu q^{-1} = S\mu r^{-1}\lambda r,$$

since the radii of the two circles are equal. The construction of qr is now determined as the intersection of the two planes whose equations are above written. It is the line drawn from the origin to the point marked qr.

In the diagram, α and β are the points on the spherical surface where the vectors $\mu + q\mu q^{-1}$ and $\lambda + r^{-1}\lambda r$ meet it, that is, the mid-points of the (curvilinear) chords on which they lie.

The operator $q(\ )q^{-1}$ turns μ about λ as an axis through the angle 2ϕ into the position $q\mu q^{-1}$, and $r^{-1}(\ )r$ turns λ about μ as an axis through the angle 2ψ into the position $r^{-1}\lambda r$. These are Hamilton's rotational operators in three dimensional space.

The cosines of the angles between r, qr and q, qr are respectively

$$Sqr/r = \cos\phi \text{ and } Sqr/q = \cos\psi.$$

Hence, the operators $q(\ )1$ and $1(\ )r$, applied to r and q respectively, turn r into qr through the angle ϕ, and q into qr through the angle ψ. These are Hathaway's right and left turns.‡

The construction for rq is indicated in the lower part of the figure, where $r\lambda r^{-1}$ and $q^{-1}\mu q$ take the places of $r^{-1}\lambda r$ and $q\mu q^{-1}$

* Hathaway, BULLETIN, vol. 4, p. 55.
† Tait's Treatise on Quaternions, 3d edition, p. 75.
‡ *Transactions Amer. Math. Soc.*, vol. 3, p. 51.

respectively and represents rotations about μ and λ in the reverse directions. The changes in the figure required by this reversal are obvious.

PACIFIC GROVE, CALIFORNIA,
February 23, 1905.

SHORTER NOTICES.

Introduction a la Géométrie Générale. By GEORGES LECHALAS. Paris, Gauthier-Villars. Pp. ix + 58.

THE point of view of M. Lechalas is different from that of most recent workers on the foundations of geometry. Ordinary usage at present applies the term mathematical science to any body of propositions deducible from a set of postulates. The term geometry is applied with various degrees of freedom to a large number of sciences, all however characterized by similar types of relation and modes of study. These abstract geometries are "represented concretely" in many different ways, as is illustrated for example, by the beautiful theorem of M. Barbarin :*

"Each of the three spaces, Euclidean, Lobachevskian, Riemannian, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straight lines of the three spaces."

To Lechalas, on the other hand, a geometry is a "form of externality." Under the title General Geometry he includes only the geometries of Euclid, Lobachevsky and Riemann (double elliptic or spherical geometry), thus excluding from consideration not only the current "bizarre geometries" but even the symmetric non-archimedean geometries of Hilbert and Veronese as well as the classical "single-elliptic" geometry. In no place do we find clear distinctions between metaphysical and mathematical questions, in a book where both are considered. On the contrary we find the following statement (page 16), which reads rather strangely in view of the vast number of different ways of representing an abstract science to the imagination :

* Quoted thus in a translation by G. B. Halsted of a report by P. Mansion on the non-euclidean researches of P. Barbarin, *Science*, n. s., vol. 20, No. 507 (September 16, 1904).

"Nous l'avons déjà dit, nous ne saurions accepter comme géométrie une analyse imaginaire, telle que celle qu'on obtient en remplaçant R par $R\sqrt{-1}$ dans les formules de la géométrie sphérique, car, là, la représentation par l'imagination n'est pas seulement impossible en fait, mais elle l'est absolument: cette analyse n'a plus aucun rapport avec une forme d'extériorité."

Curiously enough, the paragraph just quoted is followed by this: "Au contraire, la notion d'un espace à quatre dimensions repose sur des intuitions spatiales, et nous n'avons aucune raison de refuser d'admettre qu'elle pourrait répondre à des images, soit dans d'autres esprits, soit même dans les nôtres si notre sensibilité avait reçu une autre éducation."

The exposition of the usefulness of a four-dimensional point of view in studying a three-space is the best thing in Lechalas's book. He presents the distinctions between symmetry and congruence, etc., both for euclidean and spherical space, in a very elegant and vivid way, suggesting strongly the desirability in general of studying the space of a given type when situated in a space of higher number of dimensions of the same or another type. It is in this respect that Lechalas comes nearest to carrying out his avowed aim to present the subject in a style less "fragmentary" than that which its historical development has forced upon geometry.

His principal other means to this end is the introduction of the concept of curvature of a surface as independent of the space in which the surface is situated. This is done by means of the theorem of Gauss, that the integral curvature of a geodesic triangle is equal to the sum of the angles of the triangle diminished by two right angles. A geodesic line is, however, neither clearly defined nor clearly recognized as undefined. If the geodesic is accepted as an undefined element, such for example as the straight line in Hilbert's geometry, then the actual procedure of Lechalas (cf. his Chapter III) is not very different from the usual one of putting the assumptions about parallel lines in the form of statements about the angle-sum of triangles.

The book is intended primarily for readers "who have not yet formed a systematic conception of the three geometries." Its usefulness for this purpose is destroyed, in the opinion of the reviewer, by the polemical (though courteous) attitude adopted toward MM. Mansion and Barbarin. The controverted questions are such as hardly can arise if one takes the abstract point

of view, and they certainly have very little interest for American readers.

OSWALD VEBLEN.

Elementare Algebra. Akademische Vorlesungen für Studierende. By Professor Dr. EUGEN NETTO. Leipzig, B. G. Teubner, 1904. viii + 200 pp.

THE book under review is written in the informal and somewhat detailed style of the lecture as distinguished from a treatise, is rather generously supplied with well-chosen figures, and contains a good index but no exercises for the student to solve. It is the outgrowth, as the author tells us in the preface, of a course of lectures (entitled Einleitung in die Algebra) which he gives during the summer semester of each year in the University of Giessen.

The purpose of the book, and also of the lectures on which it is based, is two-fold: for those students who are to continue their work in mathematics, it is designed to bridge over the gap which usually exists between the algebra of the fitting school and that of the university; and to the non-mathematical student it presents, in a somewhat popular form but from a broad viewpoint, some of the more important problems and methods of algebra, a knowledge of which should have a place in a liberal education.

Equations of the first four degrees determine the main divisions of the book, and around these cluster a great variety of topics. Thus there are eight chapters with the following titles: I Equations of the first degree; II Pure quadratic equations; III General quadratic equations; IV Permutations and combinations; V Determinants. Linear equations; VI Binomial equations; VII Cubic equations; VIII Biquadratic equations. And among the many topics discussed under these headings may be mentioned: in Chapter I determinants of the second order, arithmetical progression, continued fractions, and indeterminate equations; in Chapters II and III Newton's method of approximation, periodic continued fractions, imaginary and complex numbers (including a rather full graphic treatment), probability that the roots of a random quadratic are real, simple and multiple valued functions and discriminants; with an equally richly varied selection for the other chapters, — Chapter V is particularly well done.

The book as a whole is carefully planned and well written; it broadens the student's point of view, stimulates his interest in the subject, and gives him no false notions which he will have to unlearn later. In the opinion of the reviewer it will prove itself very helpful, not only to the class of students for whom it was especially prepared, but also to teachers who are engaged in this field of work; it is especially to be commended to the attention of mathematics teachers in our own secondary schools.

Besides a few obvious misprints which have been detected in reading the book (on pages 92, 116, 120, 124, 126, 149, 152, 181, 189 and 194), the following minor criticisms may be mentioned: (1) The mechanical make-up of the book would be greatly improved by emphasizing the sectional divisions. This could be done by appropriate headings, in heavy-faced type perhaps, or by merely spacing between consecutive sections. As it now stands it requires a real effort to find the beginning of a section to which subsequent reference may have been made. (2) Of all the topics that such a book as this should contain, from the point of view of either of the two classes of students for whom it was written, it would seem that one of the most appropriate would be a detailed presentation of mathematical induction; yet although this method of proof is employed in the book, its essential character as a distinct type of mathematical reasoning is not brought out. Neither is the theorem of undetermined coefficients proved, though it also is freely used. (3) In connection with Cardan's solution of the cubic equation it is carefully pointed out that although the expressions for the roots involve imaginary elements, when all the roots are real and unequal, yet these imaginary parts cancel each other. It is a pity that the author did not go a step farther right here and show that the actual calculation of the α or β which is involved in these roots would itself require the solution of a cubic equation all of whose roots are real and unequal, and so explain why this is usually called the "irreducible case."

J. H. TANNER.

A First Course in Infinitesimal Analysis. By DANIEL A. MURRAY. Longmans, Green & Co., New York, London and Bombay, 1903. xvii + 439 pp.

IN common with Professor Murray's other books, this one contains numerous historical notes and references for collateral

reading. These add much to the value of the book. But in looking over the references the reviewer has been struck by what seems to him a notable lack — a lack however that is shared by practically all our elementary text-books. These references are for the most part to works in English, and not many of them can be called classical treatises. Now, unquestionably, a great majority of the students who use this book will pay but little attention to any of these references. A few of the brighter and more ambitious ones will follow them up and will find them well suited to their needs. But the young man who has in him the making of a fruitful mathematician needs the inspiration and guidance of the masters, and his elementary text-books should show him where these needs can be satisfied. In this country, at least, such a student is in great danger of being distracted by too many intermediaries. The precious years of youth, which is the creative period, should be used to the utmost advantage. Professor Murray, in making his references, does not seem to have had in mind this smallest, but by no means least important, class of students.

In reading Chapter X one cannot help feeling that the author would have succeeded better in his effort to connect the two ways of looking at integration if he had shown that the definite integral, considered as function of its upper limit, is the anti-differential of the integrand, instead of merely showing, as he does, that the definite integral, considered as the limit of a sum, can be obtained through the mediation of the anti-differential. An unfortunate statement is made in Article 126 in discussing integration of series. The fact that the limits of integration must, in general, be contained *within* the interval of convergence of the integrand (expressed as a power series) seems to have been overlooked, although a direct reference is made to Article 172, where there is a correct statement of the case.

The discussion of the integration of irrational expressions does not seem to be in keeping with the general spirit of the book. The rationale of the processes is not discussed. The student is merely told that certain substitutions will lead to the desired result. To be sure, many of these substitutions must, at best, be arbitrary. But the number of these should be reduced to a minimum.

We have been dwelling on those points wherein it seems to us the book might be improved. Happily there are very few

of these. The book is an unusually good one. Throughout there are excellent figures to illustrate points in the text. The figure exhibiting the reason for the ambiguity in sign of the derivative of $\sin^{-1}x$ is one that will be found especially helpful to the student. Indeterminate forms are assigned to a note at the end of the book. This seems to be far more in keeping with the relative importance of the subject than the treatment usually given in American text-books.

The chapter on infinite series is certainly an excellent one. It is more extensive than the corresponding one in many elementary books. The discussion of term-by-term differentiation and integration of infinite series is especially to be praised. The chapter on Taylor's theorem maintains the same standard. This chapter is put much later than usual.

There is a short chapter on differential equations, and an appendix in which hyperbolic functions, intrinsic equations, indeterminate forms, and applications to mechanics are discussed. After a set of questions and exercises and a table of integrals, some of the more common curves with their equations are given.

This latest and most extensive of Professor Murray's books is at the same time much the best one.

WILLIAM BENJAMIN FITE.

Elementary Algebra. By J. H. TANNER. New York, American Book Company. $x + 364$ pp.

THE present work is an attempt to solve a problem whose difficulties only those have realized who have seriously and conscientiously attempted to outline a course of instruction in elementary algebra which shall be teachable in the first place, but which on the other hand shall not constantly offend one's sense of rigor. There is a middle course here between Scylla and Charybdis; between the rigor of a work like Stolz and Gmeiner's Theoretische Arithmetik and the conventional algebras, whose authors draw their ideas from an age mathematically as remote as the age of stone and bronze.

Where does the best course lie between these grave perils? We do not know. *A priori* reasoning is of little avail here; it is a question which must be worked out by actual experience.

The present volume is a noteworthy and precious contribution in this direction. With ample knowledge of the founda-

tions of the subject, with wide experience of the needs of the pupil and above all with a rare mathematical tact which here is a *sine qua non*, Professor Tanner has written a work which has afforded us a very sincere pleasure in reading and which seems to us superior to any elementary English or American book on this subject which we have yet seen.

What strikes one most noticeably is the thoroughly scientific spirit of its author, the seriousness of his purpose and the simplicity and clearness of his exposition. We believe it must be a dull pupil who is not interested by the author's quiet but fascinating style. The work is indeed a veritable little classic in this respect. The usual course of the author is to begin each new topic with an introduction which gives the reader an idea of the discussion to follow. The new notions are then presented as simply as possible and illustrated with well chosen examples. By this means the reader is put in position to see the principles involved which are now stated precisely and accompanied by correct demonstrations. Problems are abundant and frequent recapitulations and review questions emphasize the main results acquired, and serve to retain them in his memory. Scattered through the book are a goodly number of notes touching on the nicer points and which no doubt will stimulate bright students to further thought and perhaps help the teacher over some hard places.

JAMES PIERPONT.

Annuaire pour l'An 1905, publié par le Bureau des Longitudes, Paris, Gauthier-Villars.

IN a long notice "Explication élémentaire des Marées," M. Hatt continues his article on the tides. The first part, printed in the Annuaire for 1904, consisted of an account of the forces producing the phenomena and the general effects produced. In the present part, the specialized effects, time and height of the tide at any particular place, are set forth with much detail. After explaining the "forced" periods of the various lunar and solar tides, the writer goes into the methods of harmonic analysis and synthesis, especially those of Darwin, by which the tide at any particular spot can be found. Although M. Hatt has not sacrificed clearness of exposition to brevity, there are pages here and there which the reader not previously instructed in the subject (and it is for such a reader that the

article is apparently written) would have difficulty in following, partly owing to the use of many technical terms, and partly to the sudden appearance of a mathematical formula. Nevertheless, the "Notice" forms a valuable addition to the very few elementary explanations of the tides which are at present available.

The present being an odd numbered year, the tables relative to physics and chemistry are omitted, in accordance with the general plan. Some additions have been made. The most interesting is a chapter devoted to meteorology. This mainly consists of a table giving mean maxima and minima temperature in various parts of the globe and two other tables relating to the temperature and barometric pressures at Paris. As usual, several of the other chapters, in particular those on geography, have been much improved with additional matter and more recent information.

ERNEST W. BROWN.

Diagrammes et Surfaces Thermodynamiques. Par J. W. GIBBS. Traduction de M. G. ROY. Avec une Introduction de M. B. BRUNHES. C. Naud (Scientia), Paris, 1904. 8vo. 100 pp. Price 2 Francs.

THIS little brochure forming Volume 22 of this series of scientific publications is a translation into French of the two original memoirs by J. Willard Gibbs, published in the *Transactions of the Connecticut Academy of Sciences*, Volume 2 (1873), pages 309–342 and 382–404, and accompanied by an introduction serving to interpret for the modern reader the special significance of these epoch-making memoirs, and in particular the trend of the lines of modern development at the hands of later workers in the same field.

It would be superfluous to speak here by way of review of such classic and well-known memoirs as those here translated for the benefit of the French reader. The introduction however may well merit a word, giving as it does an excellent presentation of certain collateral and introductory points, the discussion of which will be found of aid to any one reading these memoirs for the first time. A brief résumé is also given of the leading work of Duhem, Van der Waals and others who have found in these memoirs special inspiration, and have built so largely on this foundation. The introduction together with

the memoirs themselves should serve admirably to give to the reader a comprehensive idea of these classic methods in mathematical physics, and of the leading lines of development to which they have given rise. To all interested in these subjects this translation will come as a welcome publication in separate form of a pair of memoirs, somewhat inacessible to the general reader, and yet without which the work table of the student of mathematical physics or of physical chemistry can scarcely seem well furnished.

W. F. DURAND.

NOTES.

THE twelfth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Williams College, Williamstown, Mass., on Thursday and Friday, September 7–8. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary as early as August 20.

THE April number (volume 6, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "General projective theory of space curves," by E. J. WILCZYNSKI; "Sur les opérations linéaires (deuxième note)," by MAURICE FRÉCHET; Surfaces whose geodesics may be represented in the plane by parabolas," by EDWARD KASNER; "The doubly periodic solutions of Poisson's equation in two independent variables," by MAX MASON; "Definition in terms of order alone in the linear continuum and in well ordered sets," by OSWALD VEBLEN; "On the structure of hypercomplex number systems," by SAUL EPSTEEN and J. H. MACLAGAN-WEDDERBURN; "On a definition of abstract groups," by E. H. MOORE; "Note on the definitions of abstract groups and fields by sets of independent postulates," by E. V. HUNTINGTON; "Definitions of a group and a field by independent postulates," by L. E. DICKSON; "On semigroups and the general isomorphism between finite groups," by L. E. DICKSON; "A set of postulates for ordinary complex algebra," by E. V. HUNTINGTON; "On imprimitive linear homogeneous groups," by H. F. BLICHFELDT.

THE April number (volume 6, number 3) of the *Annals of Mathematics* contains: "Groups of the fundamental operations of arithmetic," by G. A. MILLER; "Linear differential equations with discontinuous coefficients," by M. BOCHER; "Some physical solutions of the general equation of the nth degree," by R. E. MORITZ; "The ballistic problem," by F. GILMAN; "A theorem concerning uniform convergence," by G. D. BIRKHOFF.

THE April number (volume 27, number 2) of the *American Journal of Mathematics* contains: "On a class of differential equations," by A. CHESSIN; "Surfaces with the same spherical representation of their lines of curvature as pseudo-spherical surfaces," by L. P. EISENHART; "On the forms of sextic scrolls having no rectilinear directrix," by V. Snyder; "Determination of the ternary modular group," by L. E. DICKSON.

AT the meeting of the London mathematical society held on March 9, the following papers were read: By Mr. H. BATEMAN, "On the Weddle quartic surface"; by Professor M. T. M. HILL, Dr. L. N. G. FILON, and Mr. H. W. CHAPMAN, "On projective relations between two planes"; by Mr. P. W. WOOD, "On the theory of perpetuants."

THE annual meeting of the Deutsche Mathematiker-Vereinigung will be held at Meran, in the Tyrol, September 24 to 30, under the presidency of Professor P. STÄCKEL. While contributions to any branch of mathematics will be welcome, special reports will be given on higher arithmetic, geometry, and the partial differential equations of mathematical physics. The meeting will be affiliated with the seventy-seventh annual meeting of German naturalists and physicians. Papers intended for this meeting should be sent to the secretary, Professor Dr. A. KRAZER, Karlsruhe, Westendstrasse 57, as soon as possible.

A regular meeting of the Association of Ohio teachers of mathematics and science was held at Ohio State University, Columbus, March 25, 1905. A general session was held in the morning, followed by sectional sessions in the afternoon. The following papers were presented in the section of mathematics: "What is to be done with the high school pupil who is unable to master demonstrative geometry? Shall students who fail only in mathematics be permitted to graduate?" by T. L.

Feeney; "How to teach rational geometry," by G. B. Halsted; "Apparatus illustrating elementary trigonometric functions," by W. H. Wilson; "Methods of teaching elementary algebra," by T. M. Focke. The Association's annual dues of one dollar and a half include a subscription to *School Science and Mathematics*. The officers elected are: President, W. McPherson, Ohio State University; vice-president, F. P. Jones, University School, Cleveland; secretary, T. E. McKinney, Marietta College.

THE Association of teachers of mathematics in the Middle States and Maryland held its second annual meeting at Teachers College, New York City, on Saturday, April 22. About fifty members were in attendance. Professor E. S. Crawley presided at the morning and afternoon sessions. The following papers were read: "The educational value of mathematics," by W. H. Metzler; "Some of the logical difficulties of modern euclidean geometry," by T. K. Brown; "Method of teaching pupils to do original work in plane geometry," by E. R. Smith. The association now includes 297 members.

THE Missouri Society of teachers of mathematics held its first annual meeting at the University of Missouri, on May 6. The following papers were announced on the preliminary programme: "Correlation of arithmetic, algebra, geometry, and trigonometry," by E. Y. Burton; "An experiment in individual instruction," by William Schuyler; "A method of teaching elementary geometry," by G. R. Dean; "The teaching of mathematics in the high school," by J. W. Withers; "Some developments in elementary algebra," by F. C. Touton.

A GENERAL conference of associations of teachers of mathematics will be held at Asbury Park, N. J., on July 5, in connection with the meeting of the National Educational Association. The purpose of the conference is to consider all common interests of the associations and, in general, questions connected with the teaching of elementary mathematics in the United States.

THE Open Court Publishing Company, of Chicago, has just issued a portfolio of twelve portraits of eminent mathematicians, edited by Professor D. E. SMITH. It includes the portraits of Descartes, Pythagoras, Archimedes, Fermat, Leonardo of Pisa, Euclid, Cardan, Leibniz, Napier, Vieta, Newton and Thales.

The portraits are prepared by a photographic process, and are issued in two forms, the first on Japanese paper, the price being $5.00, and the second on plate paper, at $3.00. The originals are from Professor Smith's large collection. With each portrait is a biographical note introducing a brief bibliography.

THE firm of Gauthier-Villars, in Paris, announces that the fourth volume of E. PICARD's Traité d'analyse and volumes 2 and 3 of H. POINCARÉ's Leçons de mécanique céleste are in the press.

B. G. TEUBNER has in press a treatise on Projective differential geometry of curves and ruled surfaces, by E. J. WILCZYNSKI.

THE firm of M. Schilling, in Halle, announces a model (series 30, number 4) of the limiting surface of a parabolic (linear) congruence. The surface is of order six, has a four-fold line and two consecutive biplanar points. Only two of the sheets through the four-fold lines are real. The surface can be generated by the motion of a variable circle. The model, which is of plaster, 68 × 38 × 7 cm., was constructed by Professor K. ZINDLER, of Innsbruck.

The following advanced courses in mathematics are announced for the year 1905–1906:

HARVARD UNIVERSITY. — By Professor J. M. PEIRCE: † Introduction to higher plane curves; The calculus of quaternions (first course); Tetrahedral coördinates of points, lines, and planes, the linear complex; † The calculus of finite differences; † Linear associative algebra. — By Professors BYERLY and B. O. PEIRCE: Trigonometric series, introduction to spherical harmonics, the potential function. — By Professor B. O. PEIRCE: Methods in mathematical physics. — By Professor OSGOOD: † Infinite series and products; † Riemann's theory of functions. — By Professor BÔCHER: † The properties of polynomials, invariants; Introductory course in the theory of functions; † Definite integrals and integral equations. — By Professor BOUTON: † The elementary theory of differential equations; Geometric transformations, with special reference to the work of Sophus Lie. — By Mr. WHITTEMORE: † Celestial mechanics; † Differential geometry. — By Dr.

COOLIDGE: Non-euclidean geometry. — By Dr. HUNTINGTON: † The fundamental concepts of mathematics.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Furthermore, courses in reading and research are offered by Professor J. M. PEIRCE, on Algebraic plane curves; by Professor BYERLY, on Recent contributions to ellipsoidal harmonic analysis; by Professor BÔCHER, on Fluctuating functions; by Professor BOUTON, on Topics in the theory of continuous groups; by Mr. WHITTEMORE, on Goursat's Cours d'analyse; and by Dr. COOLIDGE, on Topics in higher geometry. — The mathematical conference will meet as usual.

YALE UNIVERSITY. — By Professor BEEBE: Celestial mechanics, two hours. — By Professor J. PIERPONT: Elliptic functions, two hours; Functions of a real variable, two hours; Functions of a complex variable, two hours; Analytic geometry, two hours; Theory of aggregates, one hour. — By Professor P. F. SMITH: Continuous groups of transformations, two hours. — By Professor H. E. HAWKES: Algebra, two hours, Advanced algebra, two hours; Teachers course in geometry; two hours; Differential equations, one hour. — By Dr. W. A. GRANVILLE: Differential geometry, two hours. — By Dr. E. B. WILSON: Advanced calculus, two hours; Analytic mechanics, two hours; Theoretical mechanics, two hours. — By Dr. C. M. MASON: Partial differential equations, two hours; Functional equations, one hour. — By Dr. D. R. CURTISS: Harmonic analysis, two hours; Taylor's series and analytic continuation, one hour. — By MR. TAYLOR: Scientific computation, one hour.

THE various foreign universities offer courses in mathematics during the summer semester of 1905 as follows:

OXFORD UNIVERSITY (Easter and Trinity terms). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the geometry of conics, two hours; Informal instruction in geometry, one hour. — By Professor E. B. ELLIOTT: Theory of functions, three hours. — By Professor A. E. H. LOVE: Theory of sound, two hours. — By Mr. J. W. RUSSELL: Algebra of quantics, II, one hour. — By Mr. A. L. DIXON: Calculus of variations, one hour. — By Mr. J. E. CAMPBELL:

Applications of moving axes to solid geometry, one hour. — By Mr. H. T. GERRANS: Line geometry, two hours. — By A. E. JOLLIFFE: Higher analytic plane geometry, two hours. — By Mr. P. J. KIRBY: Higher plane curves, two hours. — By Mr. R. F. MCNEILE: Series and continued fractions, two hours. — By Mr. C. H. THOMPSON: Differential equations, two hours. — By Mr. C. E. HASELFOOT: Geometric optics, two hours.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Elements of higher mathematics, I, four hours; with exercises, two hours; Seminar, one hour. — By Professor E. NETTO: Algebra, four hours; Differential geometry, four hours; Seminar, one hour. — By Professor H. GRASSMANN: Ordinary differential equations, with exercises, three hours; Descriptive geometry, II, with exercises, four hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ: Integral calculus, four hours; Theory of algebraic surfaces and twisted curves, two hours; Seminar, two hours. — By Professor F. ENGEL: Theory of functions, II, four hours; Theory of transformation groups, four hours; Differential geometry, one hour; Seminar, two hours. — By Professor K. TH. VAHLEN: Analytic geometry, three hours; with exercises, two hours; Descriptive geometry, two hours.

UNIVERSITY OF MÜNSTER. — By Professor W. KILLING: Differential and integral calculus, I, four hours; with exercises, one hour; Theory of numbers, three hours; Seminar, two hours. — By Professor R. v. LILIENTHAL: Analytic geometry, I, four hours; Partial differential equations, four hours; Seminar, one hour. — By Dr. M. DEHN: Synthetic geometry, four hours; Calculus of variations, three hours.

UNIVERSITY OF ROSTOCK. — By Professor O. STAUDE: Plane analytic geometry, four hours; Analytic mechanics, four hours; Exercises in descriptive geometry, two hours.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: Selected chapters of higher synthetic geometry, three hours; Theory of forces, three hours; Seminar, two hours. — By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Calculus of variations, three hours; Seminar, two hours. — By Professor J. WELLSTEIN:

Abelian functions, four hours; Riemann surfaces, one hour. — By Professor M. DISTELI: Analytic geometry of space, three hours; with exercises, one hour; Descriptive geometry, II, with exercises, four hours. — By Professor M. SIMON: Methods of elementary arithmetic in connection with algebraic analysis, two hours; Noneuclidean geometry, one hour. — By Dr. P. EPSTEIN: Vector analysis, two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. v. BRILL: Analytic geometry of space, three hours; Theory of curvature of surfaces, four hours; with exercises, two hours. — By Professor H. v. STAHL: Lower analysis, three hours; with exercises, one hour; Higher analysis, three hours; with exercises, one hour. — By Professor L. MAURER: Theory of functions, three hours; with exercises, one hour; Descriptive geometry, two hours.

UNIVERSITY OF BASEL. — By Professor H. KINKELIN: Differential and integral calculus, II, three hours; Differential equations, three hours; Plane analytic geometry, three hours. — By Professor O. SPIESS: Introduction to the theory of functions, two hours; Algebra, three hours.

UNIVERSITY OF LAUSANNE. — By Professor M. AMSTERIS: Differential and integral calculus, five hours; with exercises, two hours; Theory of elliptic functions, three hours; Definite integrals and infinite series, two hours. — By Professor H. JOLY: Analytic geometry, II, two hours; Descriptive geometry, II, six hours; Theory of algebraic curves, two hours. — By Professor O. MAYOR: Rational mechanics, I, four hours; with exercises, one hour; Rational mechanics, II, two hours — By Dr. C. JACCOTTET: Theory of potential, two hours.

ACADEMY OF NEUCHÂTEL. — By Professor L. ISELY; Applications of the calculus, three hours; Analytic geometry of quadric surfaces, two hours. — By Professor L. GABEREL: Determinants and quadratic forms, one hour; Mechanics, two hours.

UNIVERSITY OF ZÜRICH. — By Professor H. BURKHARDT: Differential and integral calculus, II, four hours; Elliptic functions, four hours; Seminar, two hours. — By Professor A. WEILER: Analytic geometry, with exercises, two hours; Descriptive geometry, with exercises, three hours; Mathematical geography, three hours; Political arithmetic, two hours.

THE division of mathematics and physics of the Royal Society of Naples has awarded its prize of 1000 lire to Professor E. PASCAL for his memoir on the theory of invariants of the ternary quartic with special reference to the conditions for splitting into inferior forms. The society now offers a prize of 500 lire for the best memoir on the theory of electrons and dispersion of light. Competing essays must be written in Italian, French or Latin and be received by the secretary before June 30, 1906.

A COMMITTEE has been formed to secure funds for the erection, at Bourg-en-Bresse, of a monument to J. J. LALANDE, on the occasion of the centenary of his death. Subscriptions may be sent to the treasurer, N. Huteau, 20 boulevard Victor Hugo, Bourg-en-Bresse (Ain), France.

PROFESSOR R. LEHMANN-FILHÉS, of the University of Berlin, has been decorated with the order of the red eagle of the fourth class.

PROFESSOR F. SCHILLING, of the new technical school at Danzig has been appointed professor of geometry at the technical school of Charlottenburg-Berlin.

PROFESSOR M. CANTOR, of the University of Heidelberg, has been elected honorary member of the Royal society of Edinburg.

PROFESSOR K. ZSIGMONDY, of the technical school at Vienna, has been appointed professor of mathematics at the German technical school of Prague.

DR. KARL STÖCHL has been appointed professor of mathematics and physics at Passau.

DR. C. CARATHÉODORY has been appointed docent in mathematics at the University of Göttingen.

PROFESSORS W. F. MEYER and A. SCHOENFLIES, of the University of Königsberg, have been made corresponding members of the Royal society of sciences at Liège.

DR. JAROLIMEK has been appointed docent in geometry at the Bohemian technical school at Brünn.

AT the recent convocation of the University of Edinburg, the honorary degree of doctor of laws was conferred upon

Professor G. A. GIBSON, of the Glasgow and West of Scotland technical college.

DR. EDWARD KASNER has been promoted to an instructorship in mathematics in Barnard College, Columbia University.

MR. W. H. ROEVER has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR C. G. ROCKWOOD, of Princeton University, has retired from active service.

PROFESSOR E. W. BROWN will spend the summer in South Africa as a member of the official party of the British association for the advancement of science, which has arranged for an extended tour through the South African colonies in connection with the meetings at Cape Town and Johannesburg. During Professor Brown's absence, from June 15 to October 30, all correspondence relating to the *Transactions* should be addressed to one of the other members of the Editorial Committee.

PROFESSOR E. B. VAN VLECK, of Wesleyan University, has received leave of absence and will spend the coming year abroad.

PROFESSOR L. W. DOWLING, of the University of Wisconsin, has been granted leave of absence for the coming academic year, which he will spend mainly in Italy.

BLOOMFIELD J. MILLER, vice-president and actuary of the Mutual Benefit Insurance Company of Newark, died in that city on April 11. Mr. Miller had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1891.

DR. DAVID MURRAY, professor of mathematics at Princeton University 1863 to 1873, and for several years adviser of the imperial minister of education of Japan, died March 21, at the age of 75 years.

RECENT catalogues of second-hand mathematical works: Henry Sotheran and Company, 37 Piccadilly, London W., catalogue 650, 1687 titles of mathematical, physical and chemical books; Gustav Fock, Schlossgasse 7–9, Leipzig, catalogue of the most important sets of journals and encyclopedias in the field of exact science, about 850 titles; A. Hermann, rue de la

Sorbonne 6, Paris V, catalogue 83, part 1, 2430 titles in mathematics, letters A to H, and 170 sets of mathematical periodicals; Nicola Zanichelli, Bologna, catalogue 29, 278 titles in mathematics and physics; K. F. Kochler, 6 Kurprinzstrasse, Leipzig, Antiquariats-Katalog 564, 3004 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL (P.). Éléments d'analyse mathématique à l'usage des ingénieurs et des physiciens. 2e édition. Paris, Gauthier-Villars, 1905. 8vo. 7 + 714 pp. Fr. 24.00

AGUGLIA (G.). La curva Φ_r^x, relativa ad un sistema lineare ∞^x e le sue applicazioni ad una teoria sintetica delle curve polari. Memoria di geometria pura. Palermo, 1904. 8vo. 75 pp. Fr. 3.00

BASSETT (A. B.). An elementary treatise on cubic and quartic curves. New York, Macmillan, 1905. 8vo. 16 + 255 pp. Cloth. $4.25

BATH (W.). Zur Theorie der gleichzeitig-hyperbolischen Schnitte der Oberflächen 2. Ordnung. (Diss.) Rostock, Leopold, 1904. 8vo. 190 pp. M. 1.50

DARBOUX (G.). A study of the development of geometric methods. Translated by G. B. Halsted. 8vo. 23 pp. (*Popular Science Monthly*, March, 1905).

FLEET (R. R.). Ueber die Quadraturen von Integralen partieller Differentialgleichungen. (Diss.) Heidelberg, 1904. 8vo. 36 pp.

GARBIERI (G.). Teoria dei determinanti. Torino, 1904. 8vo. 32 pp. Fr. 2.00

GULDBERG (A.). Sur les communs multiples des expressions linéaires aux différences finies. 1905. 8vo. 6 pp. (*Rendiconti del Circolo matematico di Palermo*, Vol 29.)

——. Ueber lineare homogene Differenzengleichungen derselben Art. Warsaw, Sikorski, 1905. 8vo. 9 pp. (*Prace matematyczno-fizyczne*, Vol. 16.)

——. Ueber lineare homogene Differenzengleichungen. Leipzig, Teubner, 1905. (*Archiv der Mathematik und Physik* (3) 8.) 8vo. 4 pp.

HALSTED (G. B.). See DARBOUX (G.).

HERMITE (C.). See LAGUERRE (E.).

JOLY (C. J.). Manual of quaternions. London, Macmillan, 1905. 8vo. 348 pp. Cloth. 10s.

JUEL (C.). Stereometri og analytisk geometri til brug i tekniske skoler. Udgivet paa foranstaltning of det tekniske Selskabs Skole. Kopenhagen, 1895. 8vo. 84 pp. Kr. 2.00

LAGUERRE (E.). Oeuvres, publiées sous les auspices de l'Académie des Sciences, par C. Hermite, H. Poincaré et E. Rouché. Vol. II: Géométrie. Paris, Gauthier-Villars, 1905. 8vo. Fr. 22.00

LINDELÖF (E.). Le calcul des résidus et ses applications à la théorie des fonctions. Paris, Gauthier-Villars, 1905. 8vo. 4 + 144 pp. Fr. 3.50

LINDEMANN (F.). Lehren und Lernen in der Mathematik. Rede beim Antritt des Rektorats der Ludwig-Maximilians-Universität, gehalten am 26 November 1904. München, Wolf, 1904. 4to. 32 pp.

LIASSE (A.). Géométries imaginaires et géométrie positive. Paris, De Rudeval, 1905. 8vo. 72 pp. Fr. 1.50

MÜLLER (F.). Karl Schellbach: Rückblick auf sein wissenschaftliches Leben. Nebst zwei Schriften aus seinem Nachlass und Briefen von Jacobi, Joachimsthal und Weierstrass. Mit. einem Bildnis Karl Schellbachs. Leipzig, Teubner, 1905. 8vo. 86 pp.

MUNDÍ Y GIRÓ (S.). Lecciones de geometría analítica. 3a edición corregida y considerablemente aumentada. Barcelona, 1904. 8vo. 576 pp. Fr. 20.00

NAUENBERG (J.). Die konforme Abbildung eines Flächenstückes, das von einer algebraischen Kurve 2nter Ordnung begrenzt wird. (Diss.) Erlangen, 1904. 8vo. 50 pp.

PICARD (E.). Sur le développement de l'analyse et ses rapports avec diverses sciences. Conférences faites en Amérique. Paris, Gauthier-Villars, 1905. 8vo. 4 + 168 pp. Fr. 3.50

—— et SIMART (G.). Théorie des fonctions algébriques de deux variables indépendantes. Vol. II, fascicule II. Paris, 1904. Pp. 207–385.

PIERCE (A. B.). The necessary and sufficient conditions under which two linear homogeneous differential equations have integrals in common. 1904. 4to. (*Annals of Mathematics* (2) 6, pp. 17–29.)

PINCHERLE (S.). Lezioni di analisi algebrica date nella reale università di Bologna e redatte per uso degli studenti. Fasc. 1. Bologna, Zanichelli, 1905. 8vo. Pp. 1–143. Fr. 4.00

POINCARÉ (H.). See LAGUERRE (E.).

RÉPERTOIRE BIBLIOGRAPHIQUE des sciences mathématiques. Publié par la Commission permanente internationale du répertoire. Série 14: Fiches 1301–1400. Paris, 1904. 12mo. Chaque Série Fr. 2.25

ROUCHÉ (E.). See LAGUERRE (E.).

SCHUR (F.). Johann Heinrich Lambert als Geometer. Rektoratsrede. Karlruhe, Braun, 1905. 8vo. 20 pp. M. 0.60

SIMART (G.). See PICARD (E.).

STEINHAUSER (J.). Ueber Enhanced Lines. (Diss.) Bonn, 1904. 8vo. 32 pp.

TORROJA Y CABALLI (E.). Teoría geométrica de las líneas Mabeadas y de las superficies desarrollables. Madrid, Fortanet, 1904. 8vo. 588 pp.

II. ELEMENTARY MATHEMATICS.

BURG (R.). Sammlung algebraischer Aufgaben für gewerbliche und technische Lehranstalten, nebst einer Abhandlung über das Stabrechnen. Frankfurt a. M., Auffarth, 1905. 8vo. M. 3.50

FOERING (H. A.). See LAMBERT (P. A.).

GIRNDT (M.). Raumlehre für Baugewerkschulen und verwandte gewerbliche Lehranstalten. 2te Auflage. Teil II: Körperlehre und Dreiecksberechnung. Leipzig, 1904. 8vo. 76 pp. M. 1.40

KREUSCHMER. See LACKEMANN (C.).

LACKEMANN (C.). Die Elemente der Arithmetik. Ein Lehrbuch für den arithmetischen Unterricht an 6-klassigen höheren Lehranstalten. 4te, vermehrte, verbesserte Auflage, unter Berücksichtigung der preussischen Lehrpläne vom Juni, 1901, bearbeitet von Dr. Kreuschmer. Breslau, Hirt, 1905. 8vo. 72 pp. M. 1.00

LAMBERT (P. A.) and FOERING (H. A.). Plane and spherical trigonometry. New York, Macmillan, 1905. 12mo. 8 + 104 pp. Cloth. $0.60

LONDON UNIVERSITY matriculation mathematics papers. Set from Jan. 1893 to Jan. 1905. (University tutorial series.) London, Clive, 1905. 8vo. 134 pp. 2s.

MAYER (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, sowie für das Selbststudium. 16. Heft. Leipzig, Schaefer, 1905. 8vo.

MUSMACHER (C.). Lehrbuch der Geometrie für Mittelschulen. Leipzig, Renger, 1905. 8vo. 3 + 58 pp. M. 0.80

NANNEI (E.). Elementi di geometria. 2a edizione. Parte I: Planimetria. Puntata II. Milano, Vallardi, 1905. 16mo.

NESTLER (A.). Der logarithmische Rechenschieber und sein Gebrauch. System Mannheim, Rietz, Perry, Nestlers Universal- und Präcisionsrechenschieber. Lahr, 1904. 8vo. 56 pp. M. 1.00

ORTEGA RODÉS (J.). Solución de la cuadratura del círculo. Barcelona, 1904. 8vo. 36 pp. (Dos problemas célebres.)

PLEBANI (B.). Il latus rectum della spirale archimedèa scoperto dall'autore ed applicato alla quadratura geometrica del cerchio. Memoria. Torino, Paravia, 1905. 8vo. 55 pp., 4 plates.

SHUTTS (G. C.). Plane and solid geometry; suggestive method. Chicago, Atkinson, 1905. 12mo. 9 + 389 pp. Cloth. $1.25

STONE (J. C.). Method in geometry. Boston, Heath, 1904. 40 pp. (Heath's mathematical monographs.) $0.10

VERESHCHAGIN (I.). Questions and exercises in plane trigonometry. St. Petersburg, 1905. 8vo. 287 pp. (Russian.) M. 4.50

III. APPLIED MATHEMATICS.

BLONDIN (J.). See PÉCHEUX (H.).

BÖTTCHER (H.). Verteilung der Elektrizität auf einem aus zwei verschiedenen Halbkugeln zusammengesetzten Konduktor. (Diss.) Leipzig, 1904, 8vo. 62 pp.

BURR (W. H.) and FALK (M. S.). The graphic method by influence lines for bridge and roof computations. New York, Wiley, 1905. 8vo. 9 + 253 pp. Cloth. $3.00

DUNKERLEY (S.). Mechanism. New York, Longmans, 1905. 8vo. 5 + 408 pp. Cloth. $3.00

DUQUESNAY. Résistance des matériaux. 3e édition. Paris, Gauthier-Villars, 1904. 8vo. Fr. 2.50

EBERT (H.). Magnetische Kraftfelder. Die Erscheinungen des Magnetismus, Elektromagnetismus und der Induktion, dargestellt auf Grund des Kraftlinien-Begriffes. 2te, vollkommen neubearbeitete Auflage. Leipzig, Barth, 1905. 8vo. 12 + 415 pp. Cloth. M. 8.00

FALK (M. S.). See BURR (W. H.).

GANS (R.). Einführung in die Vektoranalysis mit Anwendung auf die mathematische Physik. Leipzig, Teubner, 1905. 8vo. 8 + 100 pp.

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JAMIESON (A.). Textbook of applied mechanics and mechanical engineering. Vol. 2. 4th edition, enlarged. London, Griffin, 1905. 8vo. 810 pp. 10s. 6d.

KOENEN (M.). Grundzüge für die statische Berechnung der Beton- und Eisenbetonbauten. 2. durchgesehene Auflage. Berlin, Ernst, 1905. 8vo. 22 pp. M. 1.20

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MAYER (H.). Ueber die Abhängigkeit des Widerstandes und des Selbstinduktionskoeffizienten bei elektrischen Schwingungen von der Frequenz. (Diss.) Zürich, 1904. 8vo. 36 pp.

MERRIFIELD (W. V.). See GILL (J.).

MONTEVERDE (F.). Teoria dei sistemi diottrici; raffronto fra la teoria analitica e quella di geometria elementare. Chiavari, 1904. 8vo. 88 pp., 5 plates. Fr. 1.50

MOYER (J. A.). Descriptive geometry for students of engineering. 2d edition. New York, Wiley, 1905. 8vo. 8 + 198 pp. Cloth. $2.00

MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. Vol. 1. 4te, vermehrte Auflage. Stuttgart, Kröner, 1905. 8vo. 7 + 576 pp. M. 18.00

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NECULCÉA (E.). See PÉCHEUX (H.).

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PÉCHEUX (H.). Traité théorique et pratique d'électricité. Avec notes additionnelles de J. Blondin et E. Neculcéa. Paris, 1904. 8vo. 720 pp. Fr. 18.00

POINCARÉ (H.). Leçons de mécanique céleste. Vol. I: Théorie générale des perturbations planétaires. Paris, Gauthier-Villars, 1905. 8vo. 6 + 367 pp. Fr. 12.00

RÖTTINGER (M.). See WALTHER (K.).

SCHÜSSLER (R.). Orthogonale Axonometrie. Ein Lehrbuch zum Selbststudium. Leipzig, Teubner, 1905. 8 + 170 pp.

SPIJKER (N.). Der Körper grösster Anziehung eines Ellipsoides. (Diss.). Zürich, 1904. 8vo. 85 pp.

VOS (M. DE). Leerboek der lagere geodesie. Groningen, 1905. 8vo. 12 + 494 pp. Fl. 8.50

WALTHER (K.) und RÖTTINGER (M.). Technische Wärmelehre (Thermodynamik). Leipzig, Göschen, 1905. 16mo. 44 pp. M. 0.80

WICKERSHEIMER (E.). Les principes de la mécanique. Paris, Dunod, 1905. 8vo. 132 pp. Fr. 4.00

WILKENS (A.). Untersuchungen über Poincarésche periodische Lösungen des Problems der drei Körper. 1905. 29 pp. M. 2.00